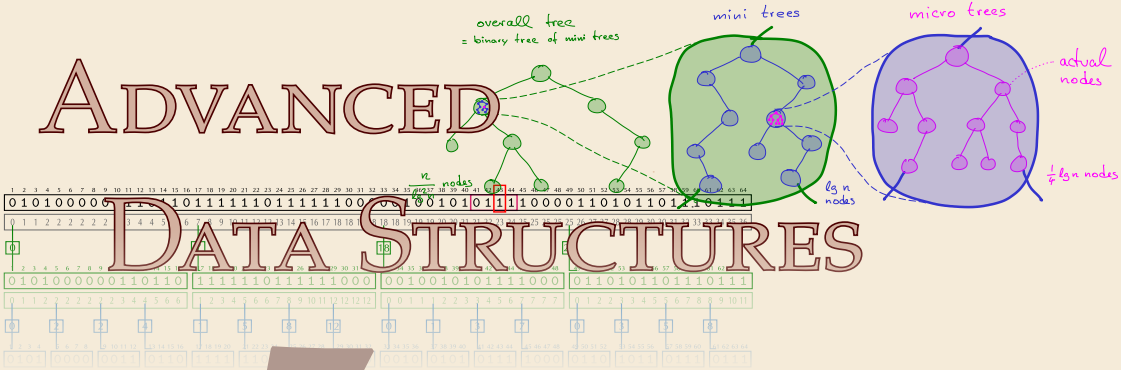

Advanced Data Structures

5 Integer Data Structures

Advanced Data Structures · Summer 2026

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CS650 (Summer 2026)
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Outline

5 Integer Data Structures

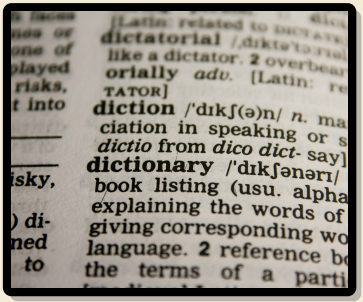
5.1 Integer Dictionaries

Recall: Symbol table ADT

Java: `java.util.Map<K,V>`, C++: `std::(unordered_)map`

Symbol table / Dictionary / Map / Associative array / key-value store:

Python dict `{k:v}`



- `put(`k`,`v`)` *Python* dict: `d[`k`] = `v
  Put key-value pair (k, v) into table
- `get(`k`)` *Python* dict: `d[`k`]`
  Return value associated with key k
- `delete(`k`)` *Python* dict: `del d[`k`]`
  Remove key k (any associated value) form table
- `contains(`k`)` *Python* dict: k `in d`
  Returns whether the table has a value for key k
- `isEmpty(), size(), create()`

Recall: Symbol table ADT

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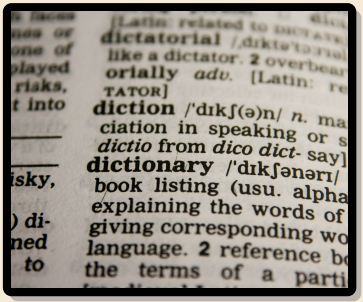

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Unlike before: Focus here on unordered case

- no rank-based queries
- values uninteresting

⇝ summarize `get` and `put` as `lookup(`k`)`

Integer Dictionaries

For unsorted dictionaries, best solutions based on *hashing*.

- Whatever the actual objects we store, always apply a hash function first

⇝ Consider here directly (dynamic) **sets of integers**

Hash Tables

Notation

- **universe** $U = [0..u)$ of allowed values
- **universe size** $u = |U|$, usually $u = 2^w$
- $S \subseteq U$ dynamic **set** of integers stored in our dictionary
- $n = |S|$ current **size** of set, $S = \{x_1, \ldots, x_n\}$

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- use **hash table** $T[0..m)$ with m **buckets**
- **hash function** $h : U \to [0..m)$

$\rightsquigarrow$ element $x \in U$ stored in $T[h(x)]$

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  - linked list $\rightsquigarrow$ *chaining hashing*
  - sequence of positions in T $\rightsquigarrow$ *open-addressing hashing*

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- in general: $T[h(x)]$ may be a **secondary data structure**
  - linked list $\rightsquigarrow$ *chaining hashing*
  - sequence of positions in T $\rightsquigarrow$ *open-addressing hashing*
- if $h(x) = h(y)$, we have a **collision** (between x and y)

Hashing must be randomized

Collisions are inevitable

- usually want S represented with $O(n)$ space
  - $\rightsquigarrow$ $m = O(n)$
- typically thus $|U| = 2^w \gg m$
  - $\rightsquigarrow$ h must have many collisions

$\rightsquigarrow$ For any *fixed* hash function h, worst-case set S has many collisions

$\rightsquigarrow$ cannot have meaningful worst-case time bounds (even if typical case good)

365 ≈ 25 people to get 50% collision for birthdays

$n \approx 2\sqrt{m}$

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Randomized Hashing

- draw *random* $h \in \mathcal{H}$
  - $\rightsquigarrow$ running time expected over random choice, worst-case w.r.t. S
- allowing all $\mathcal{H} = \{h \mid h : U \to [0..m)\}$ requires ~~m^u~~ space $\log(m^u) = u \cdot \log m$
- $\rightsquigarrow$ must choose sufficiently small $\mathcal{H}$

Universal Hashing

Many guarantees of varying strength studied for $\mathcal{H}$ . . . here only simplest needed

Definition 5.1 (Universal Family)

Let $\mathcal{H}$ be a set of hash functions from U to $[m]$ and $|U| \geq m$.
Assume $h \in \mathcal{H}$ is chosen uniformly at random.

(a) Then $\mathcal{H}$ is called a *c-universal* if

$$\forall x_1, x_2 \in U \;:\; x_1 \neq x_2 \implies \mathbb{P}\big[h(x_1) = h(x_2)\big] \leq \frac{c}{m}.$$

for fully random h

$h(x_1) \overset{D}{=} \mathcal{U}([m])$

$h(x_2) \overset{D}{=} \mathcal{U}([m])$

$\Rightarrow \mathbb{P}[h(x_1) = h(x_2)] = \frac{1}{m}$

(b) $\mathcal{H}$ is called *strongly universal* or *pairwise independent* if

$$\forall x_1, x_2 \in U, y_1, y_2 \in R \;:\; x_1 \neq x_2 \implies \mathbb{P}\big[h(x_1) = y_1 \wedge h(x_2) = y_2\big] \leq \frac{1}{m^2}.$$

◀

Examples of universal families

Universal families

- $\mathcal{H}_1 = \{h_{ab} : a \in [1..p), b \in [0..p)\}$ $(c = 2)$
- $\mathcal{H}_0 = \{h_{ab} : a \in [0..p), b \in [0..p)\}$ $(c = 4)$
- $\mathcal{H}_2 = \{h_a : a \in [1..2^k), a \text{ odd}\}$ $(c = 2)$

$h_{ab}(x) = \left(a \cdot x + b \bmod p\right) \bmod m$ $\quad p$ prime, $p \geq m$

$h_a(x) = \left(a \cdot x \bmod 2^k\right) \text{ div } 2^{k-\ell}$ $\quad u = 2^k, m = 2^\ell$

What does universality buy us?

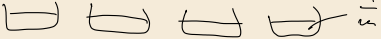
Lemma 5.2 (Universal collisions)

Consider hashing n keys $x_1, \ldots, x_n$ into m buckets using a random hash function h chosen from a c-universal family of hash functions $\mathcal{H}$. Let C denote the number of pairwise collisions, i. e., $C = \left|\{(i,j) : 1 \le i < j \le n \wedge h(x_i) = h(x_j)\}\right|$.

Then $\mathbb{E}[C] < \frac{cn^2}{2m}$.

$C \le \binom{n}{2}$ $\quad C \ge \binom{\frac{n}{m}}{2} \cdot m \simeq \frac{n^2}{m^2} \cdot m$

Proof $\mathop{\mathbb{E}}_{h}[C] = \sum_{1 \le i < j \le n} \mathop{\mathbb{E}}_{h}[C_{ij}] \le \binom{n}{2} \frac{c}{m} < \frac{cn^2}{2m}$



$C_{ij} = [h(x_i) = h(x_j)]$ $\quad \mathop{\mathbb{E}}_{h}[C_{ij}] = \mathop{\mathbb{P}}_{h}[h(x_i) = h(x_j)] \le \frac{c}{m}$

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Then $\mathbb{E}[C] < \frac{cn^2}{2m}$.

Lemma 5.3 (Universal bin size)

Consider the scenario of Lemma 5.2. Let $X_j = |T[j]|$ be the number of elements in bucket j.

(a) $\mathbb{E}[X_{h(x_i)}] \le 1 + c \cdot \frac{n}{m}$

(b) Then $\mathbb{P}\left[\max X_j > \sqrt{2c} \cdot \frac{n}{\sqrt{m}}\right] \le \frac{1}{2}$

(a) $\mathbb{E}\left[X_{h(x_i)}\right] = 1 + \sum_{j \neq i} \mathbb{E}[C_{ij}] \le 1 + (n-1)\frac{c}{m}$

(b) Any bucket w/ $\hat{X}$ keys implies $\binom{\hat{X}}{2}$ pairwise collisions

$$\Rightarrow C \geq \binom{\hat{X}}{2} \geq \frac{(\hat{X}-1)^2}{2}$$

Note: $\hat{X} \geq \sqrt{\frac{2cn^2}{m}} + 1$ implies

$$\frac{(\hat{X}-1)^2}{2} \geq \frac{cn^2}{m} \quad \text{which implies} \quad C \geq \frac{cn^2}{m} \geq 2\,\mathbb{E}[C]$$

$\Rightarrow$ by Markov inequality $\boxed{\mathbb{P}\{X \geq a\,\mathbb{E}[X]\} \leq \frac{1}{a} \quad X \geq 0}$

$$\mathbb{P}\left\{\max X_j > \sqrt{2c} \cdot \frac{n}{\sqrt{m}}\right\} \leq \frac{1}{2}$$

5.2 Perfect Hashing

Perfect Hashing

A hash function $h : [u] \to [m]$ is called

- *perfect* for a set $S = \{x_1, \ldots, x_n\} \subset [u]$ if $i \neq j$ implies $h(x_i) \neq h(x_j)$ $\quad n \leq m$
- *minimal* for set $S = \{x_1, \ldots, x_n\} \subset [u]$ if $m = n$

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Perfect Hashing

- only possible for $n \leq m$
- stringent requirement $\rightsquigarrow$ first focus on **static** S

Perfect Hashing

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Perfect Hashing

- only possible for $n \leq m$
- stringent requirement $\rightsquigarrow$ first focus on **static** ~~$\mathcal{S}$~~ S
- further requirements
  1. Hash function must be fast to evaluate (ideally $O(1)$ time)
  2. Hash function must be small to store (ideally $O(1)$ words of space)
  3. Have small m (ideally $m = \Theta(n)$)
  4. should be fast to compute given S (ideally $O(n)$ time)

Fredman-Komlós-Szemeredi Scheme

Theorem 5.4 (FKS Static Perfect Hashing)

Given a c-universal family of hash functions $\mathcal{H}$ and a static set S, we can construct in $O(n)$ expected time a perfect hash function $h : S \to [m]$ with $m = O(n)$ and evaluation time $O(t)$ for t the time to evaluate functions in $\mathcal{H}$. ◀

(with universal families from before, all good properties fulfilled!)

Step 1: Simple, but space inefficient

What # buckets do we need to make random $h \in \mathcal{H}$ perfect with good prob.?
w/p $\geq \frac{1}{2}$

n fixed $\quad m = \#\text{buckets} = m(n)$

birthday paradox $\quad n \approx \sqrt{2m \ln 2} \approx \sqrt{m}$

Let's try $m = n^2$

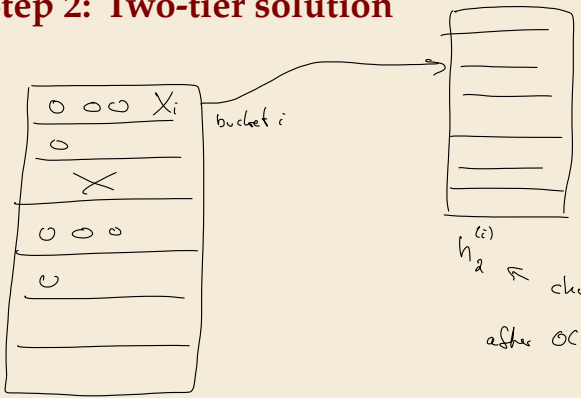
$$\mathbb{P}[\hat{X} \geq 2] \leq \mathbb{P}[C \geq 1] = \mathbb{P}\left[C \geq \frac{n^2}{m}\right] = \mathbb{P}\left[C \geq 2\,\mathbb{E}[C]\right] \leq \frac{1}{2}$$

$\hat{X}$ not perfect

$\hat{X} \geq 2 \Rightarrow C \geq 1 = \frac{n^2}{m}$

$\Rightarrow$ w/p $\geq \frac{1}{2}$ h is perfect! but w/ $m = n^2$ 😢

Step 2: Two-tier solution



pretend $c = 1$ universal

choose these with $m_i = X_i^2$

after $O(1)$ tries, each $h_2^{(i)}$ perfect

$h_1 : \mathcal{U} \to [0..m)$

$h_2^{(h_1(x))}(x)$ overall hash function

evaluation time $2 \cdot t + O(1)$

Space # buckets $= m = \sum_{i=1}^{s} m_i = \sum_{i=1}^{s} X_i^2$

(total space $O(n+m)$)

$$\mathbb{E}[m] = \sum_{i=1}^{s} \mathbb{E}[X_i^2] = \sum_{i=1}^{s} \mathbb{E}[X_i(X_i-1)] + \underbrace{\sum_{i=1}^{s} \mathbb{E}[X_i]}_{=n}$$

$x^2 = x(x-1) + x$

$$= n + 2 \underbrace{\sum_{i=1}^{s} \mathbb{E}\binom{X_i}{2}}_{= \mathbb{E}[C] = \frac{n^2}{2s}} = n + \cancel{2} \frac{n^2}{\cancel{2}s} = n + \frac{1}{2}n = O(n)$$

After $O(1)$ expected tries, find h_1 with $m \leq 3 \cdot n$

For each bucket, $O(1)$ expected tries finds perfect $h_2^{(i)}$ with $m_i = X_i^2$ slots

5.3 Dynamic Perfect Hashing

Dietzfelbinger et al. Scheme

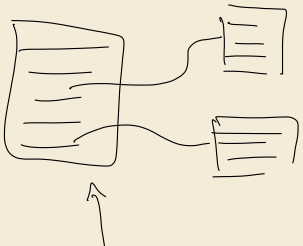
Theorem 5.5 (Dynamic Perfect Hashing)

Dynamic perfect hashing stores a subset $S \subset [0..2^w)$ of n integers in a randomized data structure supporting $O(1)$ worst-case `lookup`, $O(1)$ expected amortized time `insert` and `delete` and uses $O(n)$ words of space. ◀

Dietzfelbinger, Karlin, Mehlhorn, Meyer auf der Heide, Rohnert, Tarjan: *Dynamic Perfect Hashing: Upper and Lower Bounds*, SICOMP **1994**

Note: *This assumes that w is the word size, so that there are universal hash families with $O(1)$ evaluation time for the integers we store.*

Conceptually: FKS but w/ doubling arrays throughout.



top-level hash table $\quad \frac{8}{15}\lceil 30 \cdot M \rceil$

has s slots $\quad s = s(M) = \Theta(M)$ $\quad\quad$ $c \in (0,1)$ $\quad$ think $c = \frac{1}{2}$

target upper bound for # elements $M = (1+c) \cdot n$ initially

$$\boxed{\text{Invariant} \quad M \leq \frac{1+c}{1-c} \cdot n} \qquad \frac{3/2}{3/4} = 2n$$

$h \in \mathcal{H}_s$ universal hash function

$h : \mathcal{U} \to [0..s)$

secondary HT $\quad W_j := \{ x \in S \cdot h(x) = j \}$ $\quad b_j = |W_j|$

use # slots $\quad s_j := 2 \cdot m_j (m_j - 1)$

m_j = **bound for** b_j (since last rebuild)

$h_j \in \mathcal{H}_{s_j}$

$$\boxed{\text{Invariant} \quad \overset{(*)}{\sum_{j=0}^{s(M)-1}} s_j \leq \frac{32 M^2}{s(M)} + 4M = \Theta(M)}$$

if (*) violated ⇒ global rebuild

also: after c_N updates

deletion: use tombstones — mark as deleted, but leave in table

overwrite upon insert

remove (only) at rebuilding HT

local rebuilding: triggered upon collision

$O(1)$ tries to get new & perfect h_j

INSERT: find slot for x

- global rebuild if needed (by count)
- local rehash if collision

  if new local size violates (*) ⇒ global rebuild

  otherwise local rehash

o insert x

```
procedure Insert(x);
  count ← count + 1;
  if count > M
  then
    RehashAll(x);
  else
    j ← h(x);
    if position h_j(x) of subtable T_j contains x
      then
        if x is marked "deleted" then remove this tag;
      else (* x is new for W_j *)
        b_j ← b_j + 1;
        if b_j ≤ m_j
          then (* size of T_j sufficient *)
            if position h_j(x) of T_j is empty
              then
                store x in position h_j(x) of T_j;
              else
                go through the subtable T_j, put all elements
                not marked "deleted" into a list L_j, and
                mark all positions of T_j empty;
                append x to list L_j; b_j ← length of L_j;
                repeat h_j ← randomly chosen function in H_{s_j}
                until h_j is injective on the elements of list L_j;
                for all y on list L_j store y in position h_j(y) of T_j;
          else (* T_j is too small *)
            m_j ← 2 · max{1, m_j}; s_j ← 2m_j(m_j − 1);
            if condition (2.3) is still satisfied
              then (* double capacity of T_j *)
                allocate new space, namely s_j cells, for new subtable T_j;
                go through old subtable T_j, put all elements
                not marked "deleted" into a list L_j,
                and mark all positions empty;
                append x to list L_j; b_j ← length of L_j;
                repeat h_j ← randomly chosen function in H_{s_j}
                until h_j is injective on the elements of list L_j;
                for all y on list L_j store y in position h_j(y) of T_j;
              else (* level-1-function h "bad" *)
                RehashAll(x);
```

```
procedure RehashAll(x);
    (* RehashAll is either called by Insert with a parameter x ∈ U,
         or by Delete or Initialize without parameters. RehashAll builds a
         new table for all elements currently in the table (plus x, if given). *)
  go through the whole table T, put all elements not tagged "deleted"
  into a list L, count them, and mark all positions in T "empty";
  if x ∈ U then append x to L;
  count ← length of list L;
  M ← (1 + c) · max{count, 4};
  repeat h ← randomly chosen function in H_{s(M)};
    for all j, 0 ≤ j < s(M), do form a list L_j of all x ∈ L with h(x) = j;
    for all j, 0 ≤ j < s(M), do
      b_j ← length of list L_j; m_j ← 2 · b_j; s_j ← 2m_j(m_j − 1);
  until condition (2.3) is satisfied;
  for all j, 0 ≤ j < s(M), do
    allocate space s_j for subtable T_j;
    repeat h_j ← randomly chosen function in H_{s_j}
    until h_j is injective on the elements of list L_j;
    for all x on list L_j do store x in position h_j(x) of T_j;


procedure Delete(x);
  count ← count + 1;
  j ← h(x);
  if position h_j(x) of subtable T_j contains x
    then mark x as "deleted"
    else return(x is not a member of S);
  if count ≥ M
    then (* start new phase *)
      RehashAll();


procedure Lookup(x);
  j ← h(x);
  if position h_j(x) of subtable T_j contains x (not marked "deleted")
    then return("x is a member of S")
    else return("x is not a member of S");


procedure Initialize;
  T ← an empty table;
  RehashAll();
```

Analysis sketch

Lemma 1: Expected time before global rehash $\Theta(n)$

Lemma 2: Entire phase starting w/ global rehash w/ n elements takes $O(n)$ time total.

Local rehash: since all tables built so that twice the current # elements would have $\mathbb{P}[\text{no collisions}] \geq \frac{1}{2}$

$\Rightarrow$ hash tables last until capacity exceeded (forced rebuild)

$\Rightarrow$ $\mathbb{E}$ cost $O(1)$ amortized per update

Global rehash: similar calculation as for C shows that (*) holds for cn updates w/ $p \geq \frac{1}{2}$ $\square$.

Dynamic Perfect Hashing – Discussion

- 👍 Strongest possible guarantee for `lookup`
- 👍 All operations in $O(1)$ is clearly optimal (even though *expected amortized*)
- ⇝ (almost) perfect dictionary?

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- 👎 Main downside (of hashing generally): No sorted-dictionary operations

5.4 Binary Tries

Recall: Ordered symbol tables

On top of updates and `lookup`, we have

- `min()`, `max()`
  Return the smallest resp. largest key in the ST
- `floor(`x`)` (a.k.a. `predecessor(`x`)`) $\lfloor x \rfloor = \mathbb{Z}.$`floor(`$x$`)`
  Return largest key k in ST with $k \leq x$.
- `ceiling(`x`)` (a.k.a. `successor(`x`)`)
  Return smallest key k in ST with $k \geq x$.
- `rank(`x`)`
  Return the number of keys k in ST $k < x$.
- `select(`i`)`
  Return the ith smallest key in ST (zero-based, i. e., $i \in [0..n)$)

Sorted Dictionaries of Integers

As we will see at the end of this section, not much can be done for sorted integer dictionaries!

Fredman-Saks bound (informal version):
Maintaining a dynamic set of integers with `rank` or `select` queries
requires $\Omega(\lg n / \lg \lg n)$ time per operation.

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Surprisingly, the story shifts without `rank` and `select`!

- Consider here updates and `floor` a.k.a. `predecessor`
  - Adding `ceiling` a.k.a. `successor` trivial (by symmetry)

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- Consider here updates and `floor` a.k.a. `predecessor`
  - Adding `ceiling` a.k.a. `successor` trivial (by symmetry)
- As we will see, adding `min` / `max` trivial for considered data structures

$\rightsquigarrow$ Our following ideas also work as priority queues for integers!

- `decreaseKey` possible via `delete(`x`)` and `insert(`x'`)`

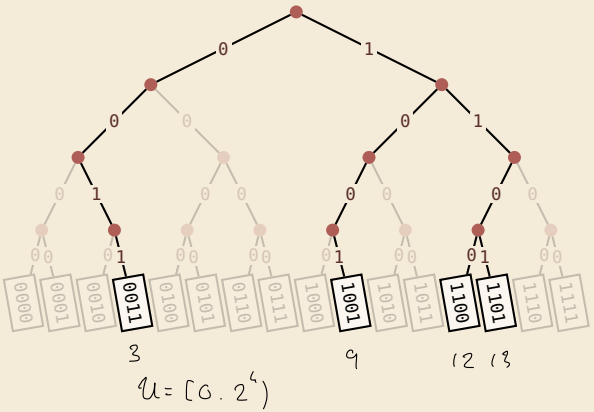
Tries

Integers are nothing but bit sequences $\rightsquigarrow$ *can use string data structures such as tries*

Tries

Integers are nothing but bit sequences ⇝ *can use string data structures such as tries*

⇝ use a binary trie



Tries

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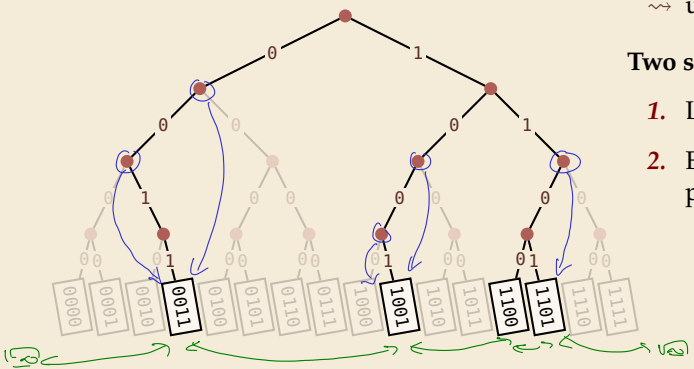

⇝ use a binary trie

Two simple additions

1. Leaves doubly linked in sorted order
2. Each nonbinary node stores *jump* pointer to extremal leaf in subtree
   - if node is missing its left child, *jump* is leftmost leaf in subtree
   - if node is missing right child, *jump* is rightmost leaf

Tries

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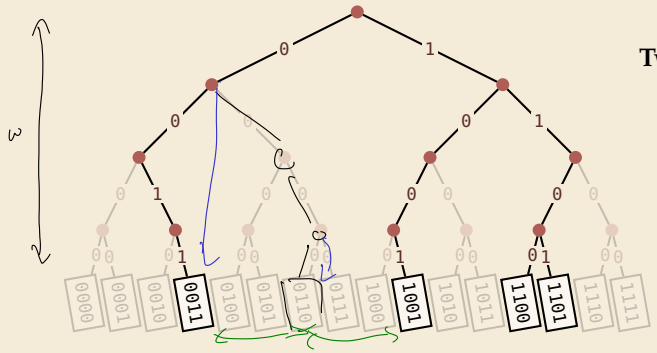

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- Additions are easy to maintain upon updates operations $O(w)$ time
- By traversing the trie an following pointers, we can implement successor/predecessor queries

Tries

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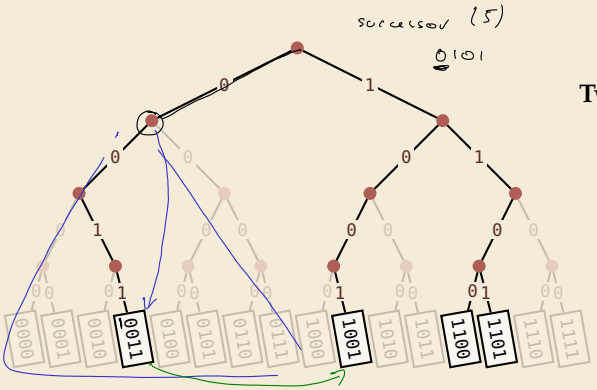

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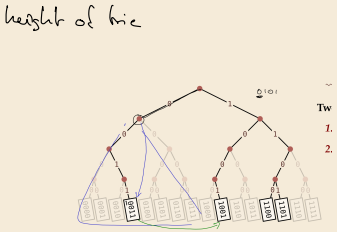
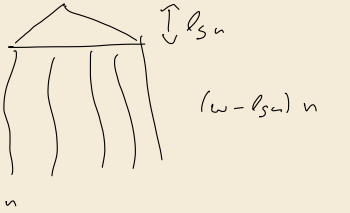
Binary Trie – Result

Theorem 5.6

A *binary trie* can store a subset $S \subseteq [0..2^w)$ of n numbers supports updates and predecessor / successor queries all in $O(w)$ time per operation for a set of w-bit integers, using $O(nw)$ bits of space.





$n \leq 2^w$

$\ell g n \leq w$

(worse than BST)

Predecessor(x)

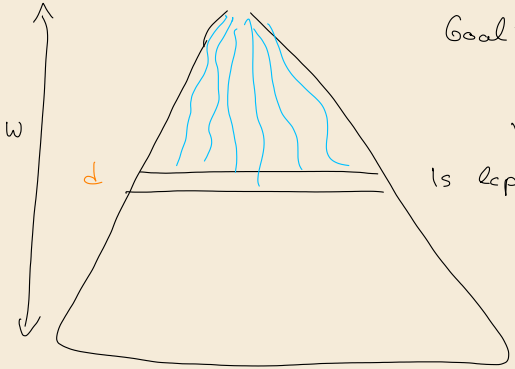
① Find longest common prefix between x and any key $y \in S$ — $O(w)$

② Follow one jump pointer and maybe one predecessor/successor pointer. — $O(1)$

5.5 x-Fast Tries

Speeding up Lookup

① Find lcp between x and S



Goal: Binary search over levels

↓ need to answer these questions.

Is lcp $\geq d$?

For each level d store set of prefixes on root to node at depth d paths

$\leq n$ d-bit integers

store these using dynamic perfect hashing!

$\Rightarrow$ can find lcp in $O(\log w)$ time!

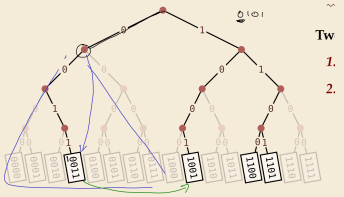
② same $\Rightarrow$ $O(\log w)$ predecessor search

x-Fast Tries

Willard

Theorem 5.7

An x-fast trie can store a subset $S \subseteq [0..2^w)$ of n integers in a randomized data structure supporting `predecessor` in $O(\lg w)$ worst case time and updates in $O(w)$ amortized expected time, using $O(nw)$ words of space.



updates: add up to w nodes

add prefix to up to

w dictionaries

$\Theta(w)$ time

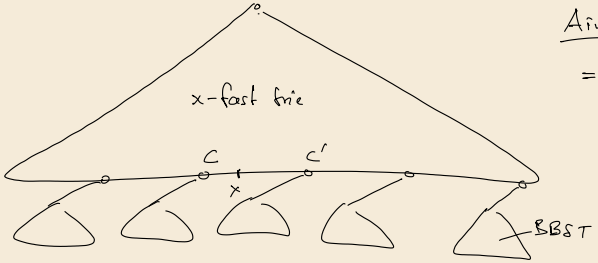
space: need to store w dictionaries w/ up to n integers each

5.6 y-Fast Tries

"Indirection" a.k.a. Don't Change Too Fast!

Idea: cluster keys into locally scalable groups

use only representatives in previous (slow-update) data structure



Aim: keep pred. search in $\Theta(\log w)$

$\Rightarrow$ $\Theta(w)$ keys per cluster

allow local search in

Pred(x): find cluster pred. C

$\Rightarrow$ search for pred in C's BST and C's successors (C')'s BST

Space: clusters $O(n)$

x-fast trie $n' = \Theta(\frac{n}{w})$

$\Rightarrow$ space $O(n'w) = O(n)$

Need to select representatives / pivots for clusters

Option 1: deterministic: split at $\geq 2w$ keys

at $< \frac{1}{2}w$, merge w/ neighbor cluster, re-split

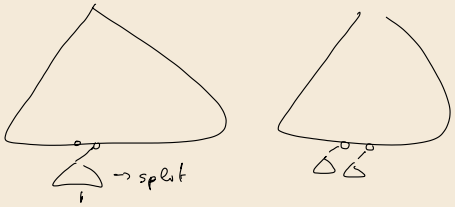
$\Rightarrow O(1)$ amortized

Option 2: every element chosen w/p $\frac{1}{w}$ as pivot (upon insert)

Represent clusters as treaps

Insert: Flip biased coin, w/ $\frac{1}{w}$ insert element into x-fast trie $O(w)$ time

split cluster



otherwise: only insert in correct cluster (found via successor)

Deletion: remove from cluster

if it was a representative, replace by predecessor from cluster $O(w)$ time

↳ w/p $\frac{1}{w}$

Updates: cost $O(\log w) + O(1)$ expected cost for x-fast trie updates

y-Fast Tries

Theorem 5.8 (y-fast tries)

A y-fast trie can store a subset $S \subseteq [0..2^w)$ of n integers in a randomized data structure supporting `predecessor` in $O(\lg w)$ ~~worst case~~ time and updates in $O(\lg w)$ amortized expected time, using $O(n)$ words of space. ◀

expected

Summary

Typical assumption: word size $\boxed{w = \Theta(\log n)}$.

Summary

Typical assumption: word size $\boxed{w = O(\log n)}$.

Consequences: (under above assumption)

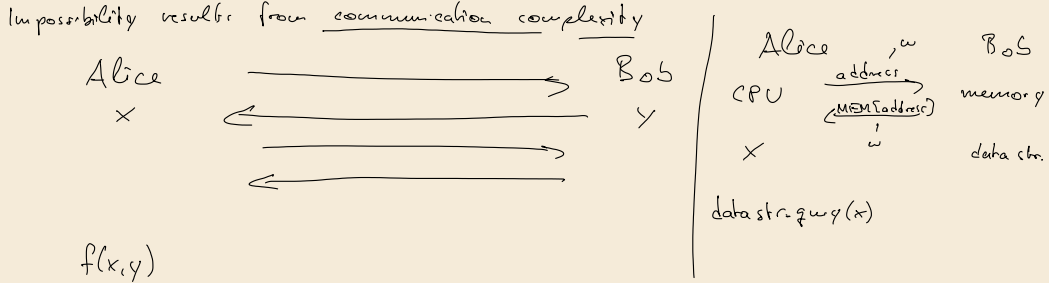
- Can maintain $S \subseteq [0..2^w)$ with the following operations in $O(\log \log n)$ amortized expected time
  - `insert`, `delete`, `lookup`
  - `predecessor`, `successor`
  - `min`, `max`
  - `decreaseKey`, `increaseKey`

5.7 Lower Bounds

The Cell-Probe Model

- model of computation mostly useful for lower bounds
- memory arranged in words of size w
- **only** cost in model is number of words of memory "touched" (read or written) (all computation for free!)

⇝ clearly, any word-RAM algorithm requires at least as much time as cell probes



Two Classical Cell-Probe Bounds [1]

Theorem 5.9 (Fredman-Saks Dynamic Rank Bound)

Maintaining a **dynamic** set $S \subseteq [0..u)$ of n integers in the cell-probe model with word size $w \leq \lg^c(u)$ (for constant c) and either operations

(a) `insert`, `delete`, and **`rank`**; or

(b) `insert`, `delete`, and **`select`**

requires **amortized** $\Omega(\log u / \log \log u)$ cell probes per operation. ◀

Fredman, Saks: *The Cell Probe Complexity of Dynamic Data Structures*, STOC **1989**

Note: since $n \leq u$, lower bound of $\Omega(\log n / \log \log n)$ holds as well.

A special case

General proof of Theorem 5.9 is technical, but let's prove (substantially) weaker version

Theorem 5.10 (Fredman Dynamic Rank Bound for $w = 1$)

Maintaining a **dynamic** set $S \subseteq [0..u)$ of n integers in the cell-probe model with word size $w = 1$ and operations `insert`, `delete`, and `rank` requires $\Omega(\log u / \log\log u)$ cell probes per operation in the **worst-case**.

Two problems in between

Dynamic Prefix Parity

maintain bitvector $A[0..u)$ initially all 0

Update : $A[i] := a$

Rank-Parity(i) $A[0] \oplus A[1] \oplus \cdots \oplus A[i] = \sum_{j=0}^{i} A[j] \bmod 2$

To reduce to dynamic rank : maintain set $S = \{i : A[i] == 1\}$

to answer rank-parity compute rank(i) mod 2

Show dynamic prefix parity need $\Omega(\lg u / \lg\lg u)$ bit probes (per op. w.c.)

Consider yet another problem:

Which-side problem maintain value $x \in [0..u)$

Set(y) $x := y$

Ask(y) is $y < x$?

obivious upper bound : $O(\lg u)$ bit probes Set store y in binary

Ask read y and compare

Lemma: Any which-side algorithm needs $\Omega(\lg u / \lg \lg_{s/t} a)$ bit probes w.c. per op.

Proof: seq. SET(i) followed only by Ask queries

Given solution with space s bits and operation t (w.c. per op)

X_i := state of memory $x_i \in \{0,1\}^s$

Hamming weight of x_i $\|x_i\|_1 \le t$ (only t cells touched)

Use $\log n$ ASK calls to binary search for i

$i \neq j \quad K_i \neq K_j$

$\Rightarrow$ lower bound for t

Lemma: Given bit $x_1, \dots, x_n \in \{0,1\}^s \quad \|x_i\|_1 \leq t$

Suppose a binary decision tree of depth d distinguishes them.

Then $n \leq \sum_{i=0}^{t} \binom{d}{i}$

$\rightsquigarrow \left| \{ x \in \{0,1\}^d : \|x\|_1 \leq t \} \right|$