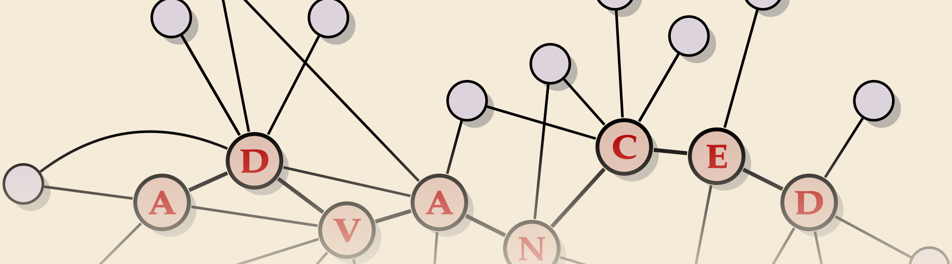

10 Approximation Algorithms

8 July 2025

Prof. Dr. Sebastian Wild

CS627 (Summer 2025)
Philipps-Universität Marburg
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Outline

10 Approximation Algorithms

10.1 Motivation and Definitions

Recap: Optimization Problems, NPO

Recall general optimization problem $U \in \mathsf{NPO}$:

- each instance x has non-empty set of *feasible solutions* $M(x)$
- objective function *cost* assigns value $cost(y)$ to all candidate solutions $y \in M(x)$
- can check in polytime
  - whether x is a valid instance
  - whether $y \in M(x)$
  - compute $cost(y) \in \mathbb{Q}$

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For each U, consider two variants:

- *optimization problem:* output $y \in M(x)$ s.t. $cost(y) = \overset{\text{min or max}}{goal}_{y' \in M(x)} cost(y')$
- *evaluation problem:* output $goal_{y \in M(x)} cost(y)$

Perfect is the enemy of good

Optimal solutions are great, but if they are too expensive to get,
maybe ***"close-to-optimal"*** suffices?

A "consistent" with problem

A ***heuristic*** is an algorithm A that always computes a feasible solution $A(x) \in M(x)$,
but we may not have any guarantees about $cost(A(x))$.

(Sometimes that's all we have . . .)

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Problem: optimal objective function value depends on x,
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Relate $cost(A(x))$ to ***OPT*** $= goal_{y \in M(x)} cost(y)$. $\rightsquigarrow$ *approximation algorithm*

Approximation Algorithms

Definition 10.1 (Approximation Ratio)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ be an optimization problem. For every $x \in L_I$ we denote its *optimal objective value* by $OPT = OPT_U(x) = goal_{y \in M(x)} cost(y)$.

Let further A be an algorithm consistent with U. $A(x) \in M(x)$

The ***approximation ratio*** $\boldsymbol{R_A(x)}$ ***of*** $\boldsymbol{A}$ ***on*** $\boldsymbol{x}$ is defined as $R_A(x) = \dfrac{cost(A(x))}{OPT_U(x)}$.

Note: For minimization problems, $R_A \geq 1$; for maximization problems $R_A \leq 1$

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Definition 10.2 (Approximation Algorithm)

An algorithm A consistent with an optimization problem $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ is called a ***c-approximation (algorithm) for U*** if

- $goal = \min$ and $\forall x \in L_I : R_A(x) \leq c$;
- $goal = \max$ and $\forall x \in L_I : R_A(x) \geq c$.

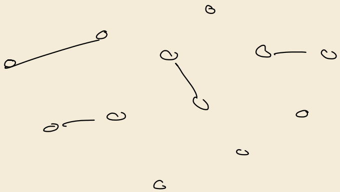
10.2 Vertex Cover and Matchings

Example: Vertex Cover

Recall the VertexCover optimization problem.
C is a VC iff $\{u, v\} \in E : \{u, v\} \cap C \neq \emptyset$
$goal = \min$
How can we vouch for a VC C to be (close to) optimal?

Need a way to lower bound OPT!

Example: Vertex Cover



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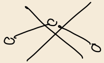
Definition 10.3 ((Maximal/Maximum/Perfect) Matching)

Given graph $G = (V, E)$, a set $M \subseteq E$ is a *matching* (in G) if (V, M) has max-degree 1.
disjoint pairs of vertices



M is *(⊆-) maxim**al*** (a.k.a. *saturated*) if no superset of M is a matching.
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M is a *perfect matching* if $|M| = |V|/2$.

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Note:

- ⊆-maximal matchings easy to find via greedy algorithm.
- Maxim**um** matchings are much more complicated, but also computable in polytime (Edmonds's "Blossom algorithm")

Matching → Vertex Cover

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If M is a matching and C is a vertex cover in G, then $|C| \geq |M|$. ◀

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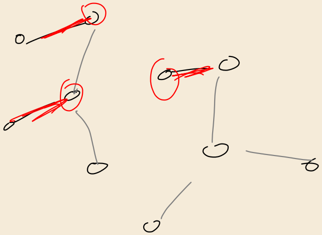
Since all $|M|$ matching edges are disjoint, C must cover them by $\geq |M|$ distinct endpoint. ∎



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```
procedure matchingVertexCoverApprox(G = (V, E))
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    for e ∈ E // arbitrary order
        if M ∪ {e} is a matching
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    return ⋃_{{u,v}∈M} {u, v}
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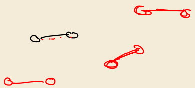
(1) C is VC

otherwise some edge uncovered



(2) 2-approx

$|C| = 2|M|$

$\frac{|C|}{OPT} \leq \frac{2|M|}{|M|} = 2$

$OPT \geq |M|$

Theorem 10.5 (Matching is 2-approx for Vertex Cover)

matchingVertexCoverApprox is a *2-approximation* for VertexCover.

Can we do better?

Maybe do smarter analysis?

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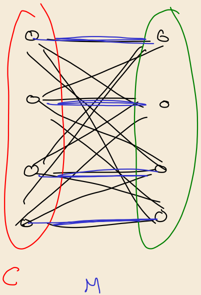
A tight example for "VC ≥ M": $K_{n,n}$



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A tight example for “VC $\geq$ M”: $K_{n,n}$

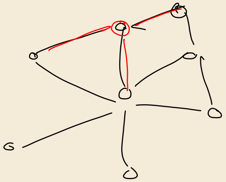
$\varepsilon > 0$ constant

Assuming the *unique games conjecture*, no polytime $(2 - \varepsilon)$ approx for VC.

Simple matching-based approximation worst-case optimal . . .

10.3 The Drosophila of Approximation: Set Cover

(Weighted) Set Cover



Definition 10.6 (SetCover)

Given: a number n, $\mathcal{S} = \{S_1, \ldots, S_k\}$ of k subsets of $U = [n]$,
and a cost function $c : S \to \mathbb{N}$.

Solutions: $\mathcal{C} \subseteq [k]$ with $\bigcup_{i \in \mathcal{C}} S_i = U$

Cost: $\sum_{i \in \mathcal{C}} c(S_i)$

Goal: min

- *cardinality version* a.k.a. UnweightedSetCover has cost $c(S) = \cancel{|S|}$ 1
- UnweightedSetCover generalizes VertexCover:
  For VertexCover instances, the sets S_i are the sets of edges incident at a vertex v
  $\rightsquigarrow$ additional property that each $e \in U$ occurs in **exactly** 2 sets S_i
- general UnweightedSetCover = Vertex Cover on hypergraphs

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We will use SetCover to illustrate various techniques for approximation algorithms.

Greedy Algorithm

Arguably simplest approach: **Greedily** pick set with current best *cost-per-new-item* ratio.

procedure greedySetCover(n, $\mathcal{S}$, c)
  $\mathcal{C} := \emptyset$; $C := \emptyset$
  // *For analysis:* $i := 1$
  while $C \neq [n]$ //assume $\bigcup S_i = [n]$
    $i^* := \arg\min_{i \in [n]} \dfrac{c(S_i)}{|S_i \setminus C|}$
    $\mathcal{C} := \mathcal{C} \cup \{i^*\}$
    $C := C \cup S_{i^*}$
    // *For analysis only:*
    // $\alpha_i := \dfrac{c(S_{i^*})}{|S_{i^*} \setminus C|}$
    // *for* $e \in S_{i^*} \setminus C$ *set* $price(e) := \alpha_i$
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Lemma 10.7 (Price Lemma)

Let $e_1, e_2, \ldots, e_n$ the order, in which greedySetCover covers the elements of U.
Then for all $j \in \{1, \ldots, n\}$ we have

$$price(e_j) \;\leq\; \frac{OPT}{n - j + 1}.$$

Proof:
Consider time when the jth element e_j is covered.

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Proof:
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$|\overline{C}| = n - (j - 1)$ elements uncovered (for $\overline{C} = U \setminus C$).
Optimal SC $\mathcal{C}^*$ covers $\overline{C}$ with cost $\leq OPT$

$$\rightsquigarrow \exists S_{i^*} : \underbrace{\frac{c(S_{i^*})}{|S_{i^*} \setminus C|}}_{\geq price(e_j)} \leq \frac{OPT}{|\overline{C}|} \leq \frac{OPT}{n - j + 1}.$$

in $\mathcal{C}^*$, but not (yet) in $\mathcal{C}$

Arbitrarily order sets in $\mathcal{C}^*$, assign prices to uncovered elements.
If all prices were $> OPT/|\overline{C}|$, covering $\overline{C}$ would cost $> OPT$. ϟ

■

Greedy Set Cover Analysis

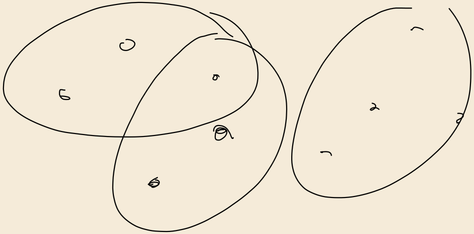
$H_n = \ln n + \overset{0.57..}{\gamma} + o(1)$

Theorem 10.8 (greedySetCover approx)

greedySetCover is an H_n-approximation for WEIGHTEDSETCOVER.

Proof:

$$c(\mathscr{C}) \;=\; \sum_{i \in \mathscr{C}} c(S_i) \;=\; \sum_{j=1}^{n} price(e_j)$$



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$$\begin{aligned} c(\mathcal{C}) &= \sum_{i \in \mathcal{C}} c(S_i) = \sum_{j=1}^{n} price(e_j) \\ &\underset{[\text{Lemma 10.7}]}{\leq} \sum_{j=1}^{n} \frac{OPT}{n-j+1} = OPT \sum_{i=1}^{n} \frac{1}{n} = H_n \cdot OPT \end{aligned}$$

$\blacksquare$

Greedy Worst Case

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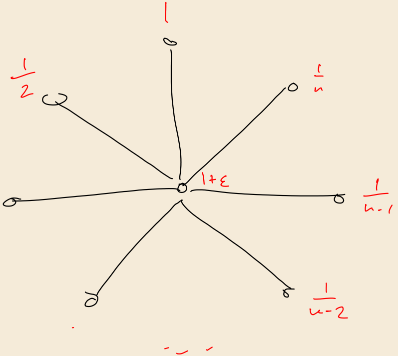
Unfortunately, bound is **tight** for greedySetCover in the worst case even on WEIGHTEDVERTEXCOVER instances:

- Consider star graph where leaves cost $\frac{1}{n}, \frac{1}{n-1}, \ldots, 1$, and middle vertex costs $1 + \varepsilon$.
- greedySetCover picks all leaves $\rightsquigarrow H_n$
- $OPT = 1 + \varepsilon$

greedy step 1

center $= \frac{1+\varepsilon}{n}$

leaf $= \frac{1}{n}$



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for greedySetCover

More complicated constructions: $\Omega(\log n)$-approx even for (UNWEIGHTED)VERTEXCOVER.

10.4 The Layering Technique for Set Cover

Size-proportional cost functions

Greedy failed on "unfair" costs for sets . . . what if costs are "nicer"?
Larger sets "should" be more costly.

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Definition 10.10 (Frequency)

The *frequency* f_e of an element $e \in [n]$ is the number of sets in which it occurs:
$f_e = |\{j : e \in S_j\}|$.
The (maximal) *frequency* of a SetCover instance is $f = \max_e f_e$. ◀

Note: (Weighted)VertexCover instance $\rightsquigarrow$ $f = 2$

Size-proportional indeed easier

Lemma 10.11 (size-proportionality → trivial f-approx)

For a size proportional weight function c we have $c(\mathcal{S}) \le f \cdot OPT$.

Proof:

$$c(\mathcal{S}) = \sum_{i=1}^{k} c(S_i) = p \sum_{i=1}^{k} |S_i|$$

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size-prop. $\rightsquigarrow$ $OPT \ge p \cdot n$

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Taking *all* sets gives f-approx, so certainly true for greedySetCover.

But probably not too many problem instances are that simple …

Layering Algorithm

Idea: Split cost function into sum of

- size-proportional part c_0 and
- a some residue c_1

$$c(S_i) = \underset{p \cdot |S_i|}{\underset{\parallel}{c_0(S_i)}} + \underbrace{c_1(S_i)}_{\geq 0}$$

procedure layeringSetCover($U, \mathcal{S}, c$)
    $p := \min\left\{\dfrac{c(S_j)}{|S_j|} : j \in [k]\right\}$
    $c_0(S_i) := p \cdot |S_i|$ *// size-prop. part*
    $c_1(S_i) := c(S_i) - c_0(S_i)$ // ≥ 0
    $\mathcal{C}_0 := \{j \in [k] : \underline{c_1(S_j) = 0}\}$
    $U_0 := \bigcup_{j \in \mathcal{C}_0} S_j$ *// covered by size-prop.*
    if $U_0 == U$
        return $\mathcal{C}_0$
    else
        $U_1 := U \setminus U_0$ *// rest of universe*
        $\mathcal{S}_1 := \{S \in \{S_1, \ldots, S_k\} \mid S \cap U_1 \neq \emptyset\}$
        $\mathcal{C}_1 :=$ layeringSetCover($U_1, \mathcal{S}_1, c_1$)
        return $\mathcal{C}_0 \cup \mathcal{C}_1$

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Theorem 10.12 (layering f-approx)

layeringSetCover is f-approx. for SetCover. ◀

Layering Algorithm

Idea: Split cost function into sum of

- size-proportional part c_0 and
- a some residue c_1

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procedure layeringSetCover($U, \mathcal{S}, c$)
    $p := \min\left\{ \frac{c(S_j)}{|S_j|} : j \in [k] \right\}$
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    $U_0 := \bigcup_{j \in \mathcal{C}_0} S_j$ *// covered by size-prop.*
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Proof:
Show by induction over recursive calls that
(a) computes cover (b) of cost $\leq f \cdot OPT$.

Basis: $U_0 = U$
All of U covered by size-prop. part ✓ (a) ✓
$\rightsquigarrow$ f-approx by Lemma 10.11 (b)

/

on restricted instance

$S' = C_0$

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Theorem 10.12 (layering f-approx)

layeringSetCover is f-approx. for SetCover.

Proof:
Show by induction over recursive calls that
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Basis: $U_0 = U$
All of U covered by size-prop. part/
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Inductive step:
IH: $\mathcal{C}_1$ covers U_1 at cost $c_1(\mathcal{C}_1) \leq f \cdot OPT(U_1, \mathcal{S}_1, c_1)$.
Let $\mathcal{C}^*$ be **optimal** set cover w.r.t. c

$c(C^*) = OPT$

Layering Algorithm

Idea: Split cost function into sum of

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Theorem 10.12 (layering f-approx)

layeringSetCover is f-approx. for SetCover.

Proof:
Show by induction over recursive calls that
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IH: $\mathcal{C}_1$ covers U_1 at cost $c_1(\mathcal{C}_1) \leq f \cdot OPT(U_1, \mathcal{S}_1, c_1)$.
Let $\mathcal{C}^*$ be **optimal** set cover w.r.t. c

size-prop.

Lemma 10.11: $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ is f-approx w.r.t. c_0.
$\rightsquigarrow$ $c_0(\mathcal{C}) \leq f \cdot c_0(\mathcal{C}^*)$ (0)

Layering Algorithm [2]

Proof (cont.):

Define $\mathcal{C}_1^* = \{i \in \mathcal{C}^* : S_i \in \mathcal{S}_1\}$

Layering Algorithm [2]

Proof (cont.):

Define $\mathcal{C}_1^* = \{i \in \mathcal{C}^* : S_i \in \mathcal{S}_1\}$

$\mathcal{C}_1^*$ is a set cover for U_1 — all S_i with $i \in \mathcal{C}^* \setminus \mathcal{C}_1^*$ only U_0

$$\rightsquigarrow \quad c_1(\mathcal{C}_1) \underset{IH}{\leq} f \cdot OPT(U_1, \mathcal{S}_1, c_1) \leq f \cdot \underline{c_1(\mathcal{C}_1^*)} \qquad (1)$$

$$OPT(U_1, S_1, c_1) \leq c_1(\mathcal{C}_1^*)$$

Layering Algorithm [2]

Proof (cont.):

Define $\mathcal{C}_1^* = \{i \in \mathcal{C}^* : S_i \in \mathcal{S}_1\}$

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$$\rightsquigarrow \quad c_0(\mathcal{C}) \leq f \cdot c_0(\mathcal{C}^*) \qquad (0)$$

$$\begin{aligned}
c(\mathcal{C}) &= c_0(\mathcal{C}) + c_1(\mathcal{C}) \\
&= c_0(\mathcal{C}) + c_1(\mathcal{C}_{\mathbf{1}}) \\
& \qquad i \in \mathcal{C}_0 \rightsquigarrow c_1 = 0 \\
&\underset{(0),(1)}{\leq} f \cdot \big(c_0(\mathcal{C}^*) + c_1(\mathcal{C}_1^*)\big) \\
&\leq f \cdot \big(c_0(\mathcal{C}^*) + c_1(\mathcal{C}^*)\big) \\
&= f \cdot \underset{\substack{\shortparallel \\ OPT}}{c(\mathcal{C}^*)}
\end{aligned}$$

∎

Layering Algorithm [2]

Proof (cont.):

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&\leq f \cdot \left(c_0(\mathcal{C}^*) + c_1(\mathcal{C}^*)\right) \\
&= f \cdot c(\mathcal{C}^*)
\end{aligned}$$

∎

Note: For VERTEXCOVER, this yields again a 2-approximation.

$\rightsquigarrow$ Same as using maximal matching

But the layering algorithm can handle *arbitrary vertex costs* (WEIGHTEDVERTEXCOVER)!

10.5 Applications of Set Cover

Shortest Superstrings

Definition 10.13 (ShortestSuperstring)

Given: alphabet Σ, set of strings $W = \{w_1, \ldots, w_n\} \subseteq \Sigma^+$

Feasible Instances: *superstrings* s of S, i. e., s contains w_i as substring for $1 \le i \le n$.

Cost: $|s|$

$s[j_i..j_i+|w_i|) = w_i$

Goal: min

Remark 10.14

Without-loss-of-generality assumption: no string is a substring of another.

Shortest Superstrings

Definition 10.13 (SHORTESTSUPERSTRING)

Given: alphabet Σ, set of strings $W = \{w_1, \ldots, w_n\} \subseteq \Sigma^+$

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Cost: $|s|$

Goal: min ◀

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- Motivation: DNA assembly (sequencing from many shorter "reads")
- General problem is NP-complete

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- Motivation: DNA assembly (sequencing from many shorter "reads")
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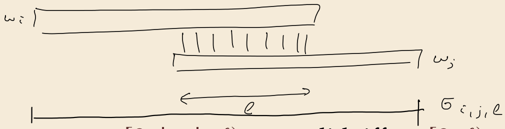
Here: Reduce this problem to SetCover!

Shortest Superstring by Set Cover

Construct *all pairwise* superstrings: overlap w_i and w_j by exactly ℓ characters (if possible)



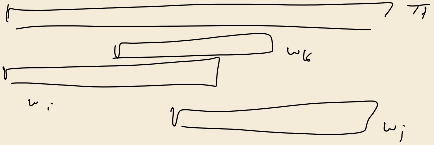
$\sigma_{i,j,\ell} = w_i[0..|w_i|-\ell) \cdot w_j$ valid iff $w_j[0..\ell) = w_i[|w_i|-\ell..|w_i|)$

$M = \left\{\sigma_{i,j,\ell} : i,j \in [u], \ell \in \left[0..\min\{|w_i|,|w_j|\}\right]\right\}$

$\rightsquigarrow$ **Set Cover instance:**

- **Universe:** $[n]$ $\quad\rightsquigarrow$ try to *cover* all words in W with superstring ...
- **Subsets:** $S = \{S_\pi : \pi \in W \cup M\}$ $\quad$ ... by combining pairwise superstrings.
  where $S_\pi = \{k \in [n] \;:\; \exists i,j : w_k = \pi[i..j)\}$
- **Cost function:** $c(S_\pi) = |\pi|$

w_k occurs as substring in π



$i, j, k \in S_\pi$

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- **Cost function:** $c(S_\pi) = |\pi|$

Given set-cover solution $\{S_{\pi_1}, \ldots, S_{\pi_k}\}$
$\rightsquigarrow$ superstring $s = \pi_1 \ldots \pi_k$ (in any order)

Shortest Superstring by Set Cover – Analysis

Lemma 10.15 (Pairwise superstrings yield 2-SC-approx)

Let W be an instance for ShortestSuperstring and (n, S, c) the corresponding SetCover instance. Let further OPT resp. OPT_{SC} be the optimal objective value of W resp. (n, S, c). Then $OPT \le OPT_{SC} \le 2 \cdot OPT$. ◀

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Corollary 10.16 ($2H_n$ approximation for superstring)

By solving the transformed set cover instance with greedySetCover, we obtain a $2H_n$-approximation for the shortest superstring problem. ◀

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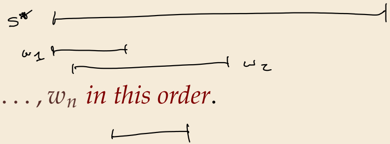
  By definition, every w_i must be contained in some π_k as a substring.

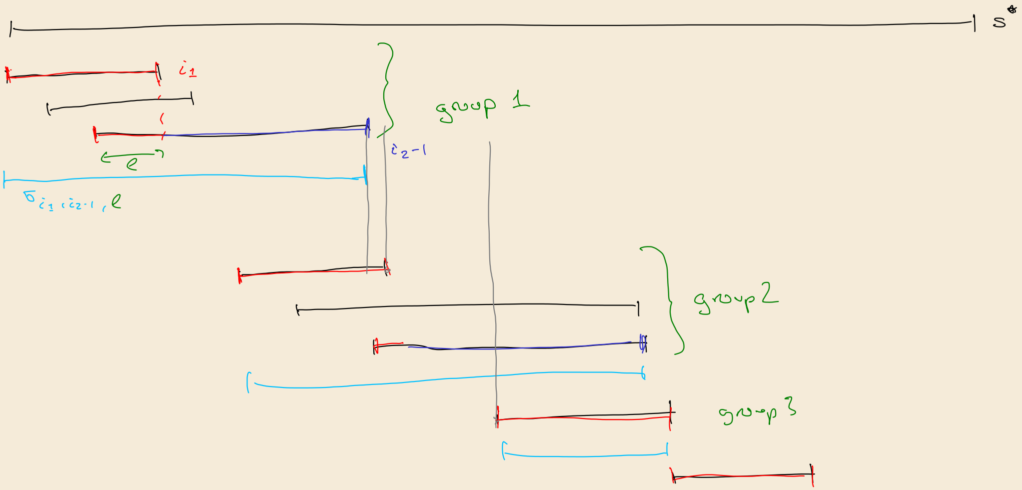
- "$OPT_{SC} \le 2 \cdot OPT$"
  $OPT = |s^*|$ for a *shortest* superstring s^* for W.
  Without loss of generality, suppose s^* contains $w_1, \ldots, w_n$ *in this order*.



Shortest Superstring by Set Cover – Analysis [2]

Proof:

Define groups: $i_1 = 1$; $i_j = \min\{i > i_{j-1} : \text{first occurrence of } w_i \text{ does not overlap } w_{i_{j-1}}\}$.



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Group j starts with w_{i_j} and ends with $w_{i_{j+1}-1}$

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(Note: Better approximation algorithms for ShortestSuperstring possible via different techniques.)

10.6 (F)PTAS: Arbitrarily Good Approximations

Approximation Schemes

The problems so far had a barrier to arbitrarily good approximations;
but sometimes we can achieve the latter!

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$\varepsilon \mapsto \mathrm{Time}_{A_\varepsilon}(c)$

FPTAS polynomial

Definition 10.17 ((F)PTAS)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, \mathbf{min})$ an optimization problem.

An algorithm $A = A_\varepsilon(x)$ with input (ε, x) is called ***polynomial-time approximation scheme (PTAS)*** for U, if for every *constant* $\varepsilon \in \mathbb{Q}_{>0}$, the algorithm A_ε is a $(1+\varepsilon)$-approximation for U with running time polynomial in $|x|$.

If the running time of $A_\varepsilon(x)$ is bounded by a polynomial in $|x|$ **and** $\boldsymbol{\varepsilon^{-1}}$, A is called a ***fully polynomial-time approximation scheme (FPTAS)*** for U. ◀

Note: PTAS could have running time $O(n^{2^{1/\varepsilon}})$ or so (akin to XP ~~running~~ XP time)

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FPTAS much stronger ... *but do they even exist for any* NP*-hard problems?* Yes!

Pseudopolynomial DP Reprise

Recall **0/1-Knapsack**: **Given:** items $1, \ldots, n$ with *weights* $w_1, \ldots, w_n$ and *values* $v_1, \ldots, v_n$;
Feasible solutions: subset of items with total weight $\leq b$
Goal: maximize total value

Approximation Idea: Work with *rounded* values (depending on ε)

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In Unit 3, we solved Knapsack

- using a DP table $V[n', b']$ = max value from items $1..n'$ and total weight $b' \le b$

$\rightsquigarrow$ $n \cdot b$ entries $\rightsquigarrow$ total time $O(n \cdot b \cdot \underline{\log(MaxInt(v))})$ $\le |x|$

$\rightsquigarrow$ good if *weights* are small, but we want to round *values*

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- actually, DP also works with values as index!

  $\boxed{\text{Assumption: } w_1, \ldots, w_n, v_1, \ldots, v_n \in \mathbb{N}}$!

- DP table $W[n', v]$ = min weight from items $\underline{1, \ldots, n'}$ with value $\underline{\underline{=v}}$

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Approximation Idea: Work with *rounded* values (depending on ε)

In Unit 3, we solved Knapsack

- using a DP table $V[n', b']$ = max value from items $1..n'$ and total weight $b' \le b$

$\rightsquigarrow$ $n \cdot b$ entries $\rightsquigarrow$ total time $O(n \cdot b \cdot \log(MaxInt(v)))$

$\rightsquigarrow$ good if *weights* are small, but we want to round *values*

- actually, DP also works with values as index!

$\boxed{\text{Assumption: } w_1, \ldots, w_n,\ v_1, \ldots, v_n \in \mathbb{N}}$

- DP table $W[n', v]$ = min weight from items $1, \ldots, n'$ with value $= v$

$$W[n', v] = \begin{cases} \min\left\{W[n'-1, v],\ \underline{W[n'-1, v - v_{n'}] + w_{n'}}\right\} & \text{if } v_{n'} < v \\ W[n'-1, v] & \text{otherwise} \end{cases}$$

(+ initial values)

Pseudopolynomial DP Reprise

Recall **0/1-Knapsack**: **Given:** items $1, \ldots, n$ with *weights* $w_1, \ldots, w_n$ and *values* $v_1, \ldots, v_n$;
Feasible solutions: subset of items with total weight $\le b$
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(+ initial values)

$\rightsquigarrow$ $n \cdot nV$ entries for $V = \max v_i$ $\rightsquigarrow$ total time $\underline{O(n^2 \cdot V \cdot \log(MaxInt(w)))}$

FPTAS for Knapsack

Convenience Assumption: any item fits in the knapsack alone, i. e., $w_i \le b$

$(1-\varepsilon)$-approx

```
procedure knapsackFPTAS(w, v, b, ε)
    V := max_{i=1,...,n} v_i
    K := εV/n
    ṽ := ⌊v/K⌋  // rounded v   w/ unit K
    return DPKnapsack(w, ṽ, b)
```

DPKnapsack is pseudopolynomial DP algorithm with running time $O(n^2 \cdot V \cdot \log(\mathit{MaxInt}(w)))$

Theorem 10.18

approxKnapsack is an FPTAS for 0/1-Knapsack.

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Proof:

First consider running time; dominated by DPKnapsack.

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Theorem 10.18

approxKnapsack is an FPTAS for 0/1-KNAPSACK.

$\tilde{V} := \max \tilde{v}_i$

Proof:

First consider running time; dominated by DPKnapsack.

$$O(n^2\tilde{V}\,\underline{\log(MaxInt(w))}) \;\leq\; O(n^2\tilde{V}|x|) \;\leq\; O\left(n^2|x|\frac{V}{K}\right) \;\leq\; O\left(n^3|x|\varepsilon^{-1}\right) \;\leq\; O\left(|x|^4\varepsilon^{-1}\right)$$

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$$O(n^2 \tilde{V} \log(MaxInt(w))) \;\le\; O(n^2 \tilde{V} |x|) \;\le\; O\left(n^2 |x| \frac{V}{K}\right) \;\le\; O\left(n^3 |x| \varepsilon^{-1}\right) \;\le\; O\left(|x|^4 \varepsilon^{-1}\right)$$

It remains to show that total value of $I = \mathrm{DPKnapsack}(w, \tilde{v}, b)$ is $v(I) \ge (1 - \varepsilon) \cdot OPT$

FPTAS for Knapsack [2]

Proof (cont.):

Let I^* be an optimal solution, $v(I^*) = \sum_{i \in I^*} v_i = OPT$

For each $i \in [n]$, we have by definition $\boxed{v_i - K \;<\; K \cdot \tilde{v}_i \;\le\; v_i \quad (*)}$.

$x - 1 \le \lfloor x \rfloor \le x$

```
procedure knapsackFPTAS(w, v, b, ε)
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Moreover, $OPT \ \geq \ V$ by our assumption that each item fits into knapsack. $\quad (V)$

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Let I^* be an optimal solution, $v(I^*) = \sum_{i \in I^*} v_i = OPT$

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Moreover, $OPT \;\ge\; V$ by our assumption that each item fits into knapsack. $\quad (V)$

We now have

$$v(I) \underset{(*)}{\ge} K \cdot \tilde{v}(I) \underset{(o)}{\ge} K \cdot \tilde{v}(I^*) \underset{(*)}{\ge} v(I^*) - nK = OPT - \varepsilon V \underset{(V)}{\ge} (1-\varepsilon) \cdot OPT$$

---

procedure knapsackFPTAS(w, v, b, ε)
  $V := \max_{i=1,\ldots,n} v_i$
  $K := \varepsilon V / n$
  $\tilde{v} := \left\lfloor \frac{v}{K} \right\rfloor$ // *rounded* v
  return DPKnapsack($w, \tilde{v}, b$)

FPTAS asks for much

Theorem 10.19 (FPTAS → FPT and pseudopolynomial)

1. $U \in \mathsf{FPTAS} \implies p\text{-}U \in \mathsf{FPT}$

2. $U \in \mathsf{FPTAS}$ and $cost(u, x) < p\big(MaxInt(x)\big)$ for some polynomial p
   $\implies \exists$ pseudopolynomial algorithm for U.

10.7 Christofides's Algorithm

Metric TSP – MST Approximation

MetricTravelingSalespersonProblem: TSP where distances obey triangle inequality

Metric TSP – MST Approximation

METRICTRAVELINGSALESPERSONPROBLEM: TSP where distances obey triangle inequality

Step 1: MST

- Consider edge-weighted complete graph $G = ([n], E, D)$ of cities with pairwise distances $D_{i,j}$.
- Compute a minimum spanning tree T in G.

Metric TSP – MST Approximation

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"Baby-Christofides": Walk around T (Euler tour after doubling all edges)
If this visits a vertex another time, simply skip it
(shortcut edge to next vertex)

Lemma 10.20

Baby-Christofides is a 2-approximation for MetricTSP.

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⇝ Walking around T uses each edge twice: cost $= 2c(T)$.

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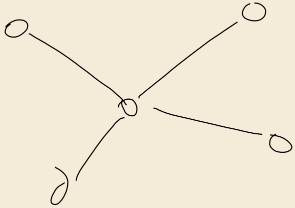
Proof:

- $\rightsquigarrow$ Walking around T uses each edge twice: cost $= 2c(T)$.
- Shortcutting does not make the tour longer by the triangle inequality.
- Removing one edge form an optimal TSP tour yields a spanning tree (path)
- $\rightsquigarrow$ $OPT \geq c(T)$.

Matchings and Tours

Can we improve upon the specific Euler tour we used?



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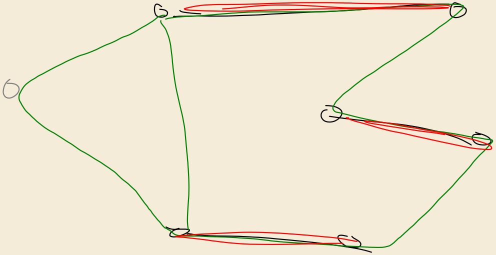
Let $V' \subseteq V$ with $|V'|$ even and let M be a minimum-cost perfect matching on V' (in the TSP graph). Then $c(M) \leq OPT/2$.



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Since $|V'|$ is even, C' is the *disjoint union of two perfect matchings* of V' (odd and even steps).

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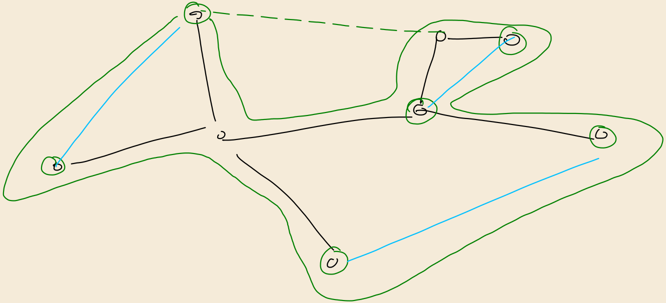
So the cheaper of these two matchings has cost $\le c(C')/2 \le c(C^*)/2 = OPT/2$.
$\rightsquigarrow$ optimal perfect matching also has $c(M) \le OPT/2$. ■

Christofides's 3/2-Approximation

Step 2: Christofides's Algorithm

- T := MST in G.
- V' := vertices with odd degree in T.
- M := minimum-cost perfect matching of V' in G.
- Output Euler cycle C in $([n], E(T) \cup M)$, shortcutting repeated vertices.



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Theorem 10.22

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Proof:

$$c(C) \quad = \quad c(T) + c(M) \quad \leq \quad OPT + OPT/2 \quad = \quad \frac{3}{2} \cdot OPT \qquad \blacksquare$$

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$$c(C) \;=\; c(T) + c(M) \;\leq\; OPT + OPT/2 \;=\; \frac{3}{2} \cdot OPT \qquad \blacksquare$$

Major open problem: Can $\frac{3}{2}$ be improved?

- Was open since 1976
- (Tiny) improvement published at STOC 2021 ($(\frac{3}{2} - \delta)$-approximation) out of PhD project of Nathan Klein (!)

10.8 Randomized Approximations

Randomized Approximation Guarantees

Definition 10.23 (Randomized δ-approx.)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, \mathbf{min})$ an optimization problem.
For $\delta > 1$, a randomized algorithm A is called *randomized δ-approximation algorithm for U*, if

- $\mathbb{P}[A(x) \in M(x)] \;=\; 1,$ (always feasible) and
- $\mathbb{P}[R_A(x) \leq \delta] \;\geq\; \frac{1}{2}$ (typically within δ)

for all $x \in L_I$. ◀

Definition 10.24 (δ-expected approx.)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, \min)$ an optimization problem.
For $\delta > 1$, a A is called *(randomized) δ-expected approximation algorithm for U*, if

- $\mathbb{P}[A(x) \in M(x)] \;=\; 1$ (always feasible) and
- $\dfrac{\mathbb{E}[cost(A(x))]}{OPT_U(x)} \;\leq\; \delta$ (expected within δ)

for all $x \in L_I$. ◀

(Minimization problems similar.)