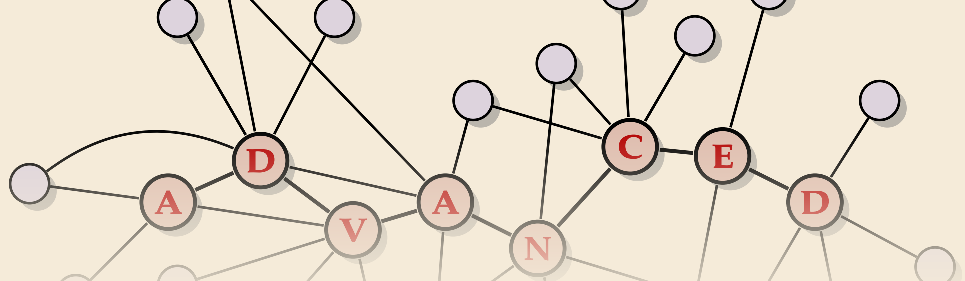

9 Random Tricks

25 June 2025

Prof. Dr. Sebastian Wild

CS627 (Summer 2025)
Philipps-Universität Marburg

version `2025-07-02 10:00`

Outline

9 Random Tricks

Uses of Randomness

- Since it is likely that $\mathsf{BPP} = \mathsf{P}$, we focus on the more fine-grained benefits of randomization:
  - simpler algorithms (with same performance)
  - improving performance (but not jumping from exponential to polytime)
  - improved robustness
- Here: Collection of examples illustrating different techniques
  - fingerprinting / hashing
  - exploiting abundance of witnesses
  - random sampling

9.1 Hashing – Balls Into Bins

Fingerprinting / Hashing

- Often have elements from huge universe $U = [0..u)$ of possible values, but only deal with few actual items $x_1, \ldots, x_n$ at one time. Think: $n \ll u$

  $\in \mathcal{U}$

- Fingerprinting can help to be more efficient in this case
  - fingerprints from $[0..m)$
  - $m \ll u$
  - *Hash Function* $h : U \to [0..m)$

h will have collisions

$(x, y \in \mathcal{U} : h(x) = h(y))$

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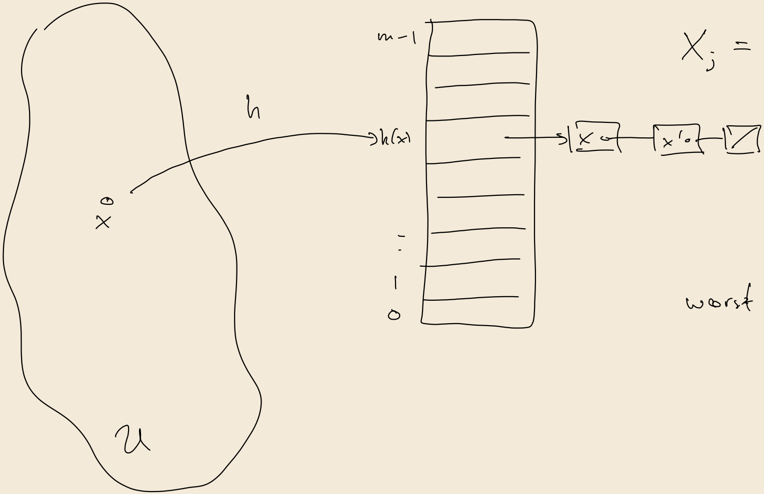
- Classic Example: hash tables and Bloom filters

Hash Tables

indirect chaining



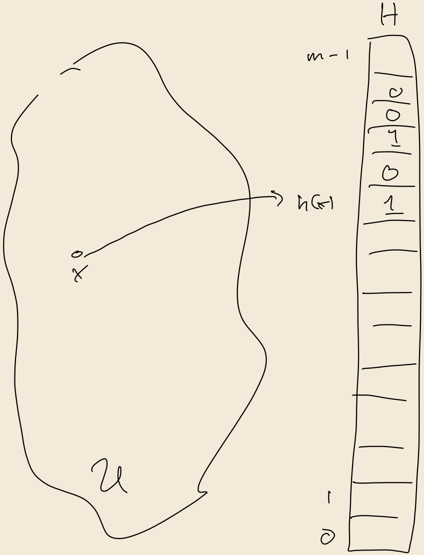
Performance:

How big are buckets?

X_j = # keys x with $h(x)=j$ in our HT

worst case $X_1 = n$

Bloom Filters



insert(x) : $H[h(x)] := 1$

query(x) : $H[h(x)]$

└ output 1 (Yes) can be a <u>false positive</u>!

output 0 (No) correct

(reduce false positive rate using independent h_1, h_2, h_3, h_4 ...)

application: segmented database

"cheap first checks"

Uniform – Universal – Perfect

Randomness is essential for hashing to make any sense! Three very different uses

1. *uniform hashing assumption*: (optimistic, often roughly right in practice!)
   How good is hashing if input is "as nicely random" as possible?

Uniform hashing assumption:

All m^n possible hash seq. $h(x_1), h(x_2), h(x_3), \ldots, h(x_n)$ are equally likely.

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   $\rightsquigarrow$ *universal hashing*: pick h at random from class H of suitable functions

   universal class of hash functions

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3. For given keys, can construct collision-free hash function
   $\rightsquigarrow$ *perfect hashing*

Uniform Hashing – Balls into Bins

Uniform Hashing Assumption:
When n elements $x_1, \ldots, x_n$ are inserted, for their *hash sequence* $h(x_1), \ldots, h(x_n)$, all m^n possible values are **equally likely**.

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$\rightsquigarrow$ might as well forget data!

Balls into bins model (a.k.a. balanced allocations)

- throw n balls into m bins

⚠ Literature usually swaps n and m!

- each ball picks bin *i.i.d. uniformly* at random

B_i = bin of ith ball $\overset{\mathcal{D}}{=} \mathcal{U}([0..m))$

- classic abstract model to study randomized algorithms
  - For hashing, effectively the best imaginable case
    tends to be a bit optimistic!
  - but: data in applications often not far from this

A Paradox?

- X_j: Number of balls in bin j:

$\rightsquigarrow$ $X_1 \stackrel{\mathcal{D}}{=} \cdots \stackrel{\mathcal{D}}{=} X_m \stackrel{\mathcal{D}}{=} \mathrm{Bin}(n, \frac{1}{m})$

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actually, just shows $X_i = n/m \pm n^{0.501}$

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  Compute counter-probability: $\mathbb{P}[\max X_j \leq 1]$

$$\underbrace{1}_{\text{ball 1}} \cdot \underbrace{\left(1 - \frac{1}{m}\right)}_{\text{ball 2}} \cdot \left(1 - \frac{2}{m}\right) \cdots \left(1 - \frac{n-1}{m}\right)$$

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  Taylor series $e^x = \underbrace{1 + x} \pm O(x^2)$ as $x \to 0$

$$1 \cdot \left(1 - \frac{1}{m}\right) \cdot \left(1 - \frac{2}{m}\right) \cdots \left(1 - \frac{n-1}{m}\right) = e^{-\frac{1}{m}} \cdot e^{-\frac{2}{m}} \cdots e^{-\frac{n-1}{m}} \cdot \left(1 \pm O\left(\left(\frac{n}{m}\right)^2\right)\right)$$

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- $\mathbb{P}[\max X_j \leq 1] = \frac{1}{2}$ for $n \approx \sqrt{2m \ln(2)}$, so for $m = 365$ days, need $n \approx 22.49$ people

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$\rightsquigarrow$ *Can't expect to see **all** bins close to **expected** occupancy.*

Fullest Bin

$$\hat{X} = \max_{j \in [m]} X_j$$

Theorem 9.1

If we throw n balls into n bins, then w.h.p., the *fullest bin* has $O\left(\frac{\log n}{\log \log n}\right)$ balls.

Proof:

$$\mathbb{P}[\max X_j \geq M] \underset{\text{union bound}}{\leq} m \cdot \mathbb{P}[X_1 \geq M]$$

$$\mathbb{P}[X_1 \geq M] = \mathbb{P}\Big[\bigcup_{\substack{I \subseteq [n] \\ |I| = M}} \text{balls } i \in I \text{ land in bin } 1\Big]$$

$$\leq \binom{n}{M} \mathbb{P}(|I| \text{ balls land in bin } 1]$$

$$\leq \binom{n}{M} \left(\frac{1}{m}\right)^M \qquad \boxed{n = m}$$

$$= \binom{n}{M} \left(\frac{1}{n}\right)^M = \frac{n!}{M! \, (n-M)! \, n^M} \leq \quad \frac{n^M}{n^M} \leq 1$$

Fullest Bin [2]

Proof (cont.):

$$\le \frac{1}{M!} \qquad \text{Stirling} \quad M! \ge \left(\frac{M}{e}\right)^M \sqrt{2\pi M}$$

$$\le \left(\frac{e}{M}\right)^M \cdot \frac{1}{\sqrt{2\pi M}} \qquad \le 1$$

$$\le \left(\frac{e}{M}\right)^M \qquad \text{need this} \le \frac{1}{n}$$

$$\left(\frac{e}{M}\right)^M \approx \frac{1}{n} \qquad M = c\,\frac{\ln n}{\ln\ln n}$$

$$\exp\left(\ln\left(\left(\frac{e}{M}\right)^M\right)\right) = \exp\left(M\left(\overset{1}{\cancel{\ln e}} - \ln M\right)\right) = \exp(-\ln n - \ln\ln\ln n - \ln\ln n) \approx \frac{1}{n}$$

to show whp. $\mathbb{P}[\hat{X} \ge M] = O(n^{-d})$

$$n\cdot\left(\frac{e}{M}\right)^M = n\left(\underbrace{\frac{e}{c}}_{<1} \cdot \frac{\ln\ln n}{\ln n}\right)^{c\frac{\ln n}{\ln\ln n}} \qquad c > e$$

$$= \exp\left(\ln n + c\cdot\frac{\ln n}{\ln\ln n}\,\ln\left(\frac{\ln\ln n}{\ln n}\right)\right)$$

$$= \exp\left(\ln n + \ln n \, \frac{c \ln\ln\ln n}{\ln\ln n} - c \ln n \cdot \cancel{\frac{\ln\ln n}{\ln\ln n}} \right)$$

$$= \exp\left((1-c)\ln n \overbrace{- \ln n + \ln n} + \ln n \, \frac{c \ln\ln\ln n}{\ln\ln n} \right)$$

$$= n^{2-c} \cdot \underbrace{\exp\left(\ln n \left(\underbrace{\frac{c \ln\ln\ln n}{\ln\ln n}}_{o(1)\, \leq\, 1 \text{ for large } n} - 1 \right)\right)}_{\leq 1 \text{ for large } n}$$

$$\leq n^{2-c} = O(n^{-d}) \quad \text{for } c > d+2$$

Fullest Bin – Consequences

- Closer analysis shows for $n = \alpha m$, constant α ("load factor"),

  $$\max X_j = \frac{\ln n}{\ln(\ln(n)/\alpha)} \cdot \big(1 + o(1)\big) \quad \text{w.h.p.}$$

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What can we learn from this?

1. Under *uniform hashing assumption*, even **worst case** of chaining hashing cost beats BST.
2. . . . but not by much.
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- Cool trick: *Power of 2 choices*
  Assume ***two*** candidate bins per ball (hash functions), take less loaded bin

$\rightsquigarrow$ $\max X_j = \ln\ln n / \ln 2 \pm O(1)$ (!) analysis more technical; details in *Mitzenmacher & Upfal*

Coupon Collector

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  - $\mathbb{E}[S] = \sum_{i=1}^{m} \mathbb{E}[S_i] = m \sum_{i=1}^{m} \frac{1}{i} = m H_m = m \ln m \pm O(m)$
- Can similarly show $\mathrm{Var}[S] = \Theta(m^2)$
  (since S_i are independent, stdev is linear + using $\mathrm{Var}[S_i] = \frac{1-p_i}{p_i^2}$)

$\rightsquigarrow$ $\sigma[S] = \Theta(m) = o(\mathbb{E}[S])$, so S converges in probability to $\mathbb{E}[S]$ (Chebyshev)

9.2 Universal Hashing

Randomized Hashing

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  - $\rightsquigarrow$ Assumes both that input is not adversarial **and** that hash functions work well.

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⇝ To replace the assumption about the input by explicit randomization, would need a *fully random hash function* $h : [n] \to [m]$

  - if we were to uniformly choose from m^n possibilities
    we'd need to store $\lg(m^n) = n \lg m$ bits just for h
  - (even if we did so, how to efficiently *evaluate* h then is unclear)
  - ϟ too expensive

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- if we were to uniformly choose from m^n possibilities
  we'd need to store $\lg(m^n) = n \lg m$ bits just for h
- (even if we did so, how to efficiently *evaluate* h then is unclear)
- ϟ too expensive

⇝ Pick h at random, but from a smaller class $\mathcal{H}$ of "convenient" functions

Universal Hashing

What's a convenient class?

Definition 9.2 (Universal Family)

Let $\mathcal{H}$ be a set of hash functions from U to $[m]$ and $|U| \geq m$.
Assume $h \in \mathcal{H}$ is chosen uniformly at random.

(a) Then $\mathcal{H}$ is called a *universal* if

$$\forall x_1, x_2 \in U \quad : \quad x_1 \neq x_2 \quad \Longrightarrow \quad \mathbb{P}_h\big[h(x_1) = h(x_2)\big] \;\leq\; \frac{1}{m}.$$

(b) $\mathcal{H}$ is called *strongly universal* or *pairwise independent* if

$$\forall x_1, x_2 \in U, y_1, y_2 \in R \quad : \quad x_1 \neq x_2 \quad \Longrightarrow \quad \mathbb{P}_h\big[h(x_1) = y_1 \wedge h(x_2) = y_2\big] \;\leq\; \frac{1}{m^2}.$$

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◀

- strong universal implies universal
- In the following, always assume (uniformly) **random** $h \in \mathcal{H}$.
- by contrast, $x_1, \ldots, x_n$ may be chosen adversarially (but all distinct) from $[u]$

Examples of universal families

$$
\begin{aligned}
h_{ab}(x) &= \Big(a \cdot x + b \bmod p\Big) \bmod m & & p \text{ prime}, p \geq m \\
h_a(x) &= \Big(a \cdot x \bmod 2^k\Big) \operatorname{div} 2^{k-\ell} & & u = 2^k, m = 2^\ell
\end{aligned}
$$

- $\mathcal{H}_1 = \{h_{ab} : a \in [1..p), b \in [0..p)\}$ is universal
- $\mathcal{H}_0 = \{h_{ab} : a \in \underline{[0}..p), b \in [0..p)\}$ is strongly universal
- $\mathcal{H}_2 = \{h_a : a \in [1..2^k), a \text{ odd}\}$ is universal

How good is universal hashing?

balls into bins

$\hat{X} \approx \frac{\ln n}{\ln \ln n}$

Theorem 9.3

Assign $x_1, \ldots, x_n \in [u]$ to bins $h(x_i) \in [m]$ using hash function h, uniformly chosen from a universal family of hash functions $\mathcal{H}$.

Let X_j be the load of bin $j \in [m]$.

Then $\mathbb{P}\left[\underbrace{\max X_j}_{\hat{X}} \geq \sqrt{2} \cdot \frac{n}{\sqrt{m}}\right] \leq \frac{1}{2}$.

$n = m \quad \sqrt{2n}$

$$\boxed{X_j \overset{\mathcal{D}}{\neq} \mathrm{Bin}(n, p)}$$

Proof:

$C_{ij} =$ 'x_i and x_j collide' $= [h(x_i) = h(x_j)]$

$\underset{\text{univ.}}{\Rightarrow} \quad \mathbb{P}[C_{ij}] \leq \frac{1}{m}$

$C = \sum_{1 \leq i < j \leq n} C_{ij} \qquad \mathbb{E}[C] = \sum \mathbb{E}[C_{ij}] \leq \binom{n}{2} \cdot \frac{1}{m} < \frac{n^2}{2m}$

$\hat{X}$ itself implies $\binom{\hat{X}}{2}$ collisions

$\Rightarrow \quad C \geq \binom{\hat{X}}{2} = \frac{\hat{X}(\hat{X}-1)}{2} \geq \frac{(\hat{X}-1)^2}{2}$

How good is universal hashing [2]

Proof:

$$\mathbb{P}\left[\hat{X} \geq n\sqrt{\frac{2}{m}} + 1\right] \leq \mathbb{P}\left[C \geq \frac{n^2}{m}\right] = \mathbb{P}\left[C \geq 2 \cdot \mathbb{E}[C]\right] \underset{\text{Markov}}{\leq} \frac{1}{2}$$

$\square$.

then $\hat{X}^2 \geq n\sqrt{\frac{2}{m}} + 1$ implies

$\frac{(\hat{X}-1)^2}{2} \geq \frac{n^2}{m}$ which implies

$C \geq \frac{n^2}{m}$

So, how good is universal hashing?

- For $n = m$, fullest bin $\leq \sqrt{2n}$
- Much worse than $\Theta(\log n / \log \log n)$!

So, how good is universal hashing?

- For $n = m$, fullest bin $\le \sqrt{2n}$
- Much worse than $\Theta(\log n / \log\log n)$!
- Note that we only proved an upper bound, however
  - bound is tight in the worst case
    (if all we know is pairwise independence of hash values)
    $\rightsquigarrow$ exercises
  - for practical choices like $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ better bounds are proven
    (close to $O(n^{1/3})$ instead of $O(n^{1/2})$)
    but still far worse than uniform hashing

9.3 Perfect Hashing

Perfect Hashing: Random Sampling

A hash function $h : [u] \to [m]$ is called

- *perfect* for a set $\mathcal{X} = \{x_1, \ldots, x_n\} \subset [u]$ if $i \neq j$ implies $h(x_i) \neq h(x_j)$
- *minimal* for set $\mathcal{X} = \{x_1, \ldots, x_n\} \subset [u]$ if $m = n$

Perfect Hashing

- only possible for $n \leq m$
- stringent requirement $\rightsquigarrow$ here focus on static $\mathcal{X}$
  - carefully chosen variants with partial rebuilding allow insertion and deletion in $O(1)$ amortized expected time

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Perfect Hashing

- only possible for $n \le m$
- stringent requirement $\rightsquigarrow$ here focus on static $\mathcal{X}$
  - carefully chosen variants with partial rebuilding allow insertion and deletion in $O(1)$ amortized expected time
- further requirements
  1. Hash function must be fast to evaluate (ideally $O(1)$ time)
  2. Hash function must be small to store (ideally $O(n)$ space)
  3. should be fast to compute given $\mathcal{X}$ (ideally $O(n)$ time)
  4. Have small m (ideally $m = \Theta(n)$)

Perfect Hashing: Simple, but space inefficient

Start simple: pick $h \in \mathcal{H}$ from universal class

what can we guarantee? n fixed

$\Rightarrow$ increase m until likely that h perfect

birthday paradox! $n \approx \sqrt{m}$

So let's try $m \geq n^2$

$\hat{X} \geq 2$ implies $C \geq 1$

which implies $C \geq \frac{n^2}{m}$

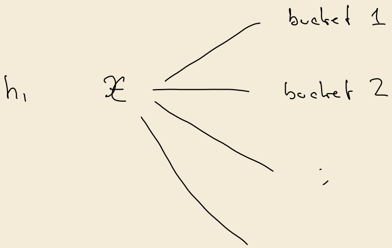
$$\mathbb{P}[\hat{X} \geq 2] \leq \mathbb{P}\left[C \geq \frac{n^2}{m}\right] = \mathbb{P}[C \geq 2\,\mathbb{E}[C]] \leq \frac{1}{2}$$

as for universal hashing above

$\Rightarrow \mathbb{P}\{\text{no collisions}\} \geq \frac{1}{2}$

good prob. to get a perfect hash function! $\lightning$ HT has n^2 space

Perfect Hashing: Two-tier solution



$O(n)$ tier 1 buckets

for each tier 1 bucket i

choose $h_2^{(i)}$ as above

using $m_i = X_i^2$ $\quad$ X_i = # elements in bucket i

↳ after expected linear time over hash function is perfect

$x \mapsto h_2^{(h_1(x))}(x)$

To show: overall space (for all secondary hash tables) small

Perfect Hashing:

1. Choose h_1 uniformly from $\mathcal{H}$ (universal) with $m = n$ bins until C (#collisions) $\leq n$

2. For each bucket $i = 1, \dots, n$
   If $X_i \geq 2$ draw random hash function $h_2^{(i)}$ from $\mathcal{H}'$ (universal) with X_i^2 bins
   repeat until $h_2^{(i)}$ perfect

3. Output $h_2^{(h_1(x))}$

X_i in bin $\Rightarrow \binom{X_i}{2}$ collisions

$$\text{"} \frac{X_i^2}{2} - \frac{X_i}{2} \text{"}$$

Claim: (a) #repetitions small

(b) space usage small

(a) ① $\mathbb{E}[C] = \binom{n}{2} \cdot \frac{1}{n} = \frac{n-1}{2} \xrightarrow{\text{Markov}} \mathbb{P}[C \leq n] \geq \frac{1}{2}$

② see above: $\mathbb{E}$ #rep. $= 2$

$\Rightarrow \mathbb{E}$ time $= O(n)$

(b) space $= \Theta(n) + \sum_{i=1}^{n} X_i^2 = \Theta(n) + \underbrace{\sum \binom{X_i}{2}}_{= C \leq n} + \underbrace{\sum \frac{X_i}{2}}_{\leq n}$

top level HT and all $h_2^{(i)}$'s

$= \Theta(n)$

9.4 Primality Testing

Abundance of Witnesses

- Suppose $L \in \mathsf{NP}$ and all of the following are true:
  - alleged certificate must be easy to check
    trivially in polytime; often very fast
  - for $x \in L$, there are **many** certificates that show $x \in L$
    not generally true, but sometimes!

$\rightsquigarrow$ Conceivable that a randomized algorithm succeeds:
  - Guess a random certificate string
  - Check if it decides the problem

Primality Testing

Testing if a given number n is *prime* is one of the oldest algorithmic questions.

Trivial approach: test for all (primes) $p \le \sqrt{n}$ whether $p \mid n$

```
procedure sieveOfEratosthenes(n):
    isPrime[2..n] := true
    for i := 2, 3, ..., ⌊√n⌋
        if isPrime[i]
            for j = i, i + 1, i + 2, ..., ⌊n/i⌋
                isPrime[i · j] := false
    return {p ∈ [2..n] : isPrime[p]}

procedure isPrimeTrivial(n):
    P := sieveOfEratosthenes(⌊√n⌋)
    return ∀p ∈ P : p ∤ n
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```

Running time:

- dominated by sieving primes up to $m = \lfloor \sqrt{n} \rfloor$
- $T(m) \;\le\; m + \sum_{\substack{p \le m \\ p \text{ prime}}} \frac{m}{p} \;\le\; m + m \sum_{p=1}^{m} \frac{1}{p}$

$\rightsquigarrow\; T(m) = O(m \log m)$

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$\rightsquigarrow$ $T(m) = O(m \log m)$

- closer analysis: actually $T(m) = O(m \log\log m)$

Space: $\sqrt{n}$ bits

Complexity of Primality Testing and Factorization

- PRIMES:
  - **Given:** Integer n in binary encoding
  - **Goal:** Check if n is a prime number
- INTEGERFACTORIZATION:
  - **Given:** Integer n in binary encoding
  - **Goal:** Find nontrivial factors $n = m_1 \cdot m_2, 2 \leq m_1, m_2 < n$ or determine "n prime"

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- If n is composite, a factorization is a certificate for *non-primality* $\rightsquigarrow$ PRIMES $\in$ co-NP
  - n encoded in binary $\rightsquigarrow$ Sieve of Eratosthenes is pseudopolynomial

$\in$ NP ?

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  - n encoded in binary $\rightsquigarrow$ Sieve of Eratosthenes is pseudopolynomial
- we will show PRIMES $\in$ co-RP $\subset$ BPP
- Major theoretical breakthrough: PRIMES $\in$ P *Agrawal, Kayal, and Saxena (2004)*
- This is not known for INTEGERFACTORIZATION
  - Indeed much of classic cryptography (RSA) builds on factoring being intractable
  - *Shor's algorithm* can factor integers on a (theoretical) quantum computer in polytime! (not clear whether or when this is a practical concern)

Does PRIMES have abundance of witnesses?

NOTPRIME

try the obvious:

factors ? composite numbers w/ 2 large prime factors ¿

hardly promising approach

since it solves FACTORIZATION

Primality Testing: Fermat's Little Theorem

Theorem 9.4 (Fermat's Little Theorem)

For p a prime and $a \in [1..p-1]$ holds

$$a^{p-1} \equiv 1 \pmod{p} \qquad (*)$$

Pick a random $a \in [1..p-1]$, compute $a^{p-1} \bmod p$ if $\neq 1$

$\Rightarrow$ p not prime.

only $\Rightarrow$, not $\Leftarrow$

indeed Carmichael numbers are not prime, but fulfill $(*)$

Primality Testing: Second Attempt

Theorem 9.5 (Euler's Criterion)

field $\mathbb{Z}$ modulo p

Let $p > 2$ an odd number.

$$p \text{ prime} \iff \forall a \in \mathbb{Z}_p \setminus \{0\} : a^{\frac{p-1}{2}} \bmod p \in \{1, \overset{\overset{p-1}{\shortparallel}}{-1}\}$$

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Theorem 9.6 (Number of Witnesses)
For every odd $n \in \mathbb{N}$, $(n-1)/2$ odd, we have:

1. If n is prime then $a^{(n-1)/2} \bmod n \in \{1, n-1\}$, for all $a \in \{1, \ldots, n-1\}$.
2. If n is not prime then $a^{(n-1)/2} \bmod n \notin \{1, n-1\}$ for *at least half* of the elements in $\{1, \ldots, n-1\}$.

Simple Solovay-Strassen Primality Test

Input: an odd number n with $(n-1)/2$ odd.

1. Choose a random $a \in \{1, 2, \ldots, n-1\}$.
2. Compute $A := a^{(n-1)/2} \bmod n$.
3. If $A \in \{1, n-1\}$ then output "n probably prime" (reject);
4. otherwise output "n not prime" (accept).

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The simple Solovay-Strassen algorithm is a polynomial **OSE-MC** algorithm to detect composite numbers n with $n \bmod 4 = 3$.

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The simple Solovay-Strassen algorithm is a polynomial **OSE-MC** algorithm to detect composite numbers n with $n \bmod 4 = 3$. ◀

Corollary 9.8

For positive integers n with $n \bmod 4 = 3$ the simple Solovay-Strassen algorithm provides a polynomial **TSE-MC** algorithm to detect prime numbers. ◀

Sampling Primes

RANDOMPRIME(ℓ, k) Input: $\ell, k \in \mathbb{N}$, $\ell \geq 3$.

1. Set X := "not found yet"; $I := 0$;
2. `while` X = "not found yet" and $I < 2\ell^2$ `do`
   - generate random bit string $a_1, a_2, \ldots, a_{\ell-2}$ and
   - compute $n := 2^{\ell-1} + \sum_{i=1}^{\ell-2} a_i \cdot 2^i + 1$

     // This way n becomes a random, odd number of length ℓ
   - Realize k independent runs of Solovay-Strassen-algorithm on n;
   - `if` at least one output says "$n \notin PRIMES$" `then` $I := I + 1$

     `else` X :="PN found"; `output` n;
3. `if` $I = 2 \cdot \ell^2$ `then output` "no PN found".

$O(\ell^5)$

Thm: RandomPrime(ℓ, ℓ) is TSE-MC, poly-time for generating primes of length ℓ.

to show (a) output prime w/ prob $\left(\frac{1}{2}\right)^k$

(b) does not fail w/ prob $\frac{1}{2\ell}$

key ingredient : prob to hit a prime number

when choosing odd n uniformly $\in [2^{\ell-1}; 2^{\ell})$

<u>Prime Number Theorem</u> : $\pi(x) = \#\text{primes} \leq x$

$$= \frac{x}{\ln(x)}(1+o(1))$$

$\rightsquigarrow$ # repetitions to sample prime $\approx \ell$

9.5 Schöning's Satisfiability

Random Sampling

If a solution is tricky to construct in a target fashion,
but many solutions are known to exist, random sampling can help.

Generate random object according to simple procedure until solution found.

We've seen ideas of random sampling in perfect hashing.

Now: Use more aggressive sampling to find rare objects.

Warmup: 2SAT

Famously, 3SAT is NP-complete.

2SAT: Given CNF formula φ with $\leq$ **2** literals per clause; is φ satisfiable?

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$a \to b \;\equiv\; \neg a \vee b$

Idea: Any clause $(\ell_1 \vee \ell_2)$ is equivalent to the *implications* $\neg\ell_1 \to \ell_2$ and $\neg\ell_2 \to \ell_1$

$\rightsquigarrow$ Represent formula as *implication graph*:

- vertices = literals in φ
- edges = all implications equivalent to some clause

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strongly connected component

- SCCs computable in linear time
- indeed, if no strong component contains contradiction, topological sort of components allows to read off satisfying assignment

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$\rightsquigarrow$ Basically, a solved problem . . . we will use it for demonstration purposes only

Warmup: A randomized 2SAT algorithm

procedure localSearch2SAT(φ, *confidence*):
    k := number of variables in ϕ
    Choose assignment $\alpha \in \{0, 1\}^k$ uniformly at random.
    for $j = 1, \ldots, confidence \cdot 2k^2$
        if α fulfills φ **return** α // *satisfiable!*
        Arbitrarily choose clause $C = \ell_1 \vee \ell_2$ not satisfied under α.
        Choose ℓ from $\{\ell_1, \ell_2\}$ uniformly at random.
        α = assignment obtained by negating ℓ.
    return `PROBABLY_NOT_SATISFIABLE`

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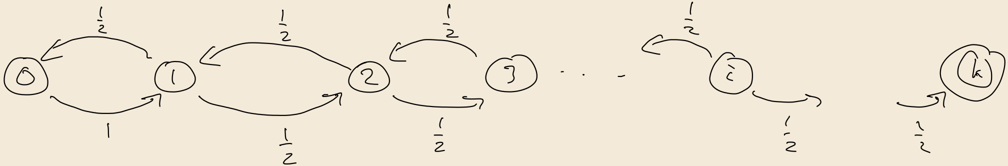
        α = assignment obtained by negating ℓ.
    return `PROBABLY_NOT_SATISFIABLE`

Theorem 9.10 (localSearch2SAT is OSE-MC for 2SAT)

Let φ be a 2SAT formula.

1. If φ is unsatisfiable, localSearch2SAT always returns `PROBABLY_NOT_SATISFIABLE`.
2. If φ is satisfiable, localSearch2SAT returns satisfying assignment with probability at least $1 - 2^{-\mathit{confidence}}$.
3. localSearch2SAT runs in $O(\mathit{confidence} \cdot k^2 n)$ time.



X_t

Y_i = starting in i expected # steps until k

Randomized 2SAT – Analysis

Proof:

Claims 1. and 3. are trivial. It remains to prove Claim 2.

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In iteration t, flip one variable in α_t to obtain α_{t+1}.

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We will analyze a *simplified* random process W that never behaves worse than localSearch2SAT
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$\rightsquigarrow$ obtain an upper bound on error probability.

φ satisfiable $\rightsquigarrow$ $\exists\, \alpha^*$ that satisfies φ.
W will stop iff $\alpha = \alpha^*$ (localSearch2SAT might stop sooner)

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$\rightsquigarrow$ obtain an upper bound on error probability.

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Randomized 2SAT – Analysis

Proof:

Claims 1. and 3. are trivial. It remains to prove Claim 2.

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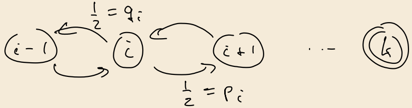
Assume W makes correct flip with prob $= \frac{1}{2}$.
$\rightsquigarrow$ $\mathbb{P}[X_{t+1} = X_t \mathbf{+1} \mid X_t] = \frac{1}{2}$ and $\mathbb{P}[X_{t+1} = X_t \mathbf{-1} \mid X_t] = \frac{1}{2}$
(except $X_t = 0$, then always +1 and $X_t = k$, then terminate)

$(X_t)_{t\geq 0}$ is thus a *Markov process*.

Randomized 2SAT – Analysis [2]



Proof (cont.):

Let now y_i be the expected number of steps to reach state k from $X = i$.

$$
\begin{aligned}
y_k &= 0 \\
y_0 &= 1 + y_1 \\
y_i &= 1 + p_i \cdot y_{i+1} + q_i \cdot y_{i-1} \qquad q_i = 1 - p_i \quad \text{for us } p_i = \frac{1}{2} \qquad 1 \le i \le k-1
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Can solve this recurrence for general p_i by writing for $i \in [1..k)$:

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$\rightsquigarrow$ Recurrence of *differences*: $\qquad \dot{y}_i = \dfrac{q_i}{p_i}\dot{y}_{i-1} - \dfrac{1}{p_i}$

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Randomized 2SAT – Analysis [3]

Proof (cont.):

Recurrences 101: Telescoping recurrence! Can solve this in full generality:

$$\rightsquigarrow \quad \boxed{\dot{y}_i \; = \; \left(\prod_{j=1}^{i} a_j\right) \cdot \dot{y}_0 + \sum_{j=1}^{i} \left(\prod_{k=j+1}^{i} a_k\right) b_j}$$

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Treat *confidence* $\cdot\, 2k^2$ iterations as *confidence* repetitions of independent attempts of $2k^2$ each.
Probability that none successful $\le 2^{-\mathit{confidence}}$. ■

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$\rightsquigarrow$ Worse than deterministic brute force!

Local Search with Restarts

- Problem first attempt: Over time, more likely to move *away* from α^*
  - Need $\approx 2^k$ expected time to move k steps closer to α^*
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  - Won't cut it for large k
- But we assume here that we start with $\neg\alpha^*$
  whereas actual random α might be (much) closer!

$\rightsquigarrow$ Keep local search for small improvements,
but restart overall method many times, to hopefully start close to α^* some time

Schöning's Randomized 3SAT Algorithm

procedure Schöning3SAT(φ, ~~confidence~~):
    k = number of variables in φ
    for $i = 1, \ldots, 24\left\lceil \sqrt{k}\left(\frac{4}{3}\right)^k \right\rceil$ **do**
        Choose assignment $\alpha \in \{0,1\}^k$ uniformly at random.
        for $j = 1, \ldots, 3k$ **do**
            if α fulfills φ **return** α
            Arbitrarily choose clause $C = \ell_1 \vee \ell_2 \vee \ell_3$ not satisfied under α.
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local search

Theorem 9.11 (Schöning3SAT is OSE-MC for 3SAT)

Let φ be a 3SAT formula with n clauses over k variables.

1. Schöning3SAT is a OSE-MC for 3SAT.
2. To be correct with probability $\geq \cancel{1 - 2^{\text{confidence}}}\ \frac{1}{2}$, it runs in time $O(\cancel{\text{confidence}} \cdot \left(\frac{4}{3}\right)^k k^{3/2} n)$

Schöning3SAT is OSE-MC for 3SAT

Proof: $1 - 2^{-\text{confidence}}$ prob.

2. follows immediately from standard OSE-MC probability amplification.

Also obvious: φ unsatisfiable $\rightsquigarrow$ Schöning3SAT returns `PROBABLY_NOT_SATISFIABLE`.

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one run of outer loop

Claim: $q \;:=\; \mathbb{P}[\text{local search finds } \alpha^*] \;\ge\; \dfrac{1}{12\sqrt{k}}\left(\dfrac{3}{4}\right)^k$

Proof:

$X = k - d_G(\alpha^*, \alpha)$ #variables correctly assigned in random α

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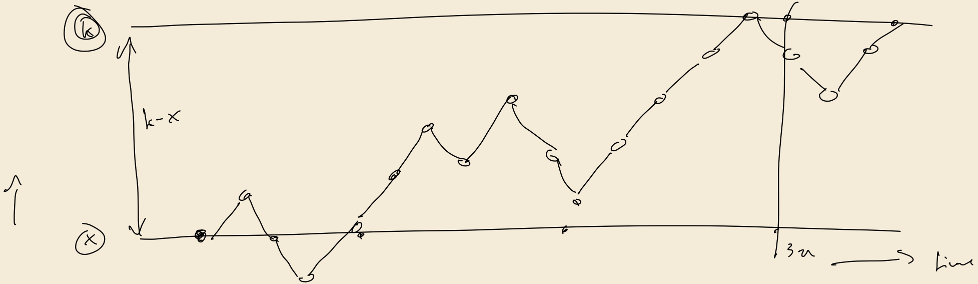

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Pessimistically, assume up-step with prob $= \frac{1}{3}$.

$$q_u \;=\; \mathbb{P}[\ge 2u \text{ up in } 3u \text{ steps}] \;\ge\; \mathbb{P}[= 2u \text{ up in } 3u \text{ steps}] \;=\; \binom{3u}{u}\left(\frac{1}{3}\right)^{2u}\left(\frac{2}{3}\right)^{u}$$

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one run of outer loop

Claim: $q \;:=\; \mathbb{P}[\text{local search finds } \alpha^*] \;\ge\; \dfrac{1}{12\sqrt{k}}\left(\dfrac{3}{4}\right)^k$

Proof:

$X = k - d_G(\alpha^*, \alpha)$ #variables correctly assigned in random α $\rightsquigarrow$ $X_0 \overset{\mathcal{D}}{=} \mathrm{Bin}(k, \frac{1}{2})$

Conditional on X, need local search to climb $u = k - X$ steps up to succeed.

We keep trying for $3k$ steps, but will only consider first $3u$ of them.

If at least $2u$ of these are up-steps, we succeed no matter which ones are up steps.

Pessimistically, assume up-step with prob $= \frac{1}{3}$.

$$q_u \;=\; \mathbb{P}[\ge 2u \text{ up in } 3u \text{ steps}] \;\ge\; \mathbb{P}[= 2u \text{ up in } 3u \text{ steps}] \;=\; \binom{3u}{u}\left(\frac{1}{3}\right)^{2u}\left(\frac{2}{3}\right)^{u}$$

Stirling-Robbins Inequality : $n! = \underbrace{e^{r_n}}_{[1,2]} \cdot \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ with $\frac{1}{12n+1} < r_n < \frac{1}{12n}$ $\rightsquigarrow$ $1 \le e^{r_n} \le 2$

Schöning3SAT is OSE-MC for 3SAT

Proof:

2. follows immediately from standard OSE-MC probability amplification.

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Schöning3SAT is OSE-MC for 3SAT [2]

Proof (Theorem 9.11 cont.):

Proof (Claim cont.):

$$q \quad = \quad \sum_{x=0}^{k} \mathbb{P}[X = x] \cdot q_{k-x}$$

Schöning3SAT is OSE-MC for 3SAT [2]

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$$q = \sum_{x=0}^{k} \mathbb{P}[X = x] \cdot q_{k-x} = \sum_{u=0}^{k} \mathbb{P}[X = k - u] \cdot q_u$$

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Expected number of independent repetitions before success: $\leq \frac{1}{q}$.

■

Schöning3SAT is OSE-MC for 3SAT [2]

Proof (Theorem 9.11 cont.):
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$$q = \sum_{x=0}^{k} \mathbb{P}[X = x] \cdot q_{k-x} = \sum_{u=0}^{k} \mathbb{P}[X = k-u] \cdot q_u \geq \frac{1}{2^k} + \sum_{u=1}^{k} \binom{k}{k-u} \left(\frac{1}{2}\right)^k \cdot q_u$$

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Expected number of independent repetitions before success: $\frac{1}{q}$.

Schöning3SAT runs $2 \cdot \frac{1}{q} = 24\sqrt{k}\left(\frac{4}{3}\right)^k$ repetitions. $\rightsquigarrow$ Success prob $\geq \frac{1}{2}$. $\blacksquare$

9.6 Karger’s Cuts

$\boxed{\text{THIS SECTION WILL BE SKIPPED AND IS NOT PART OF THE EXAM MATERIAL.}}$

Smart probability amplification: Karger's Min-Cut

Definition 9.12 (Min-Cut)

Given: A (multi)graph $G = (V, E, c)$, where $c : E \to \mathbb{N}$ is the multiplicity of an edge
Feasible Solutions: cuts of G, i. e., $M(G) = \{(V_1, V_2) \ : \ V_1 \cup V_2 = V \ \wedge \ V_1 \cap V_2 = \emptyset\}$,
Goal: Minimize
Costs: $\sum_{e \in C(V_1, V_2)} c(e)$, where $C(V_1, V_2) = \big\{\{u, v\} \in E \ : \ u \in V_1 \wedge v \in V_2\big\}$.

Random Contraction

procedure contractionMinCut($G = (V, E, c)$)
    Set $label(v) := \{v\}$ **for** every vertex $v \in V$.
    while G has more than 2 vertices
        Choose random edge $e = \{x, y\} \in E$.
        $G :=$ Contract(G, e).
        Set $label(z) := label(x) \cup label(y)$ **for** z the vertex resulting from x and y.
    Let $G = (\{u, v\}, E', c')$; **return** $(label(u), label(v))$ with cost $c'(\{u, v\})$.

Theorem 9.13 (contractionMinCut correct with some probability)

`contractionMinCut` is a polytime randomized algorithm that finds a minimal cut for a given multigraph G with n vertices with probability $\geq 2/\big(n(n-1)\big)$. ◀

Lemma 9.14 (Threshold for contractionMinCut)

Let $l : \mathbb{N} \to \mathbb{N}$ a monotonic, increasing function with $1 \leq l(n) \leq n$. If we stop `contractionMinCut` whenever G only has $l(n)$ vertices and determine for the resulting graph G/F deterministically a minimal cut, then we need time in

$$O(n^2 + l(n)^3)$$

and we find a minimal cut for G with probability at least

$$\frac{\binom{l(n)}{2}}{\binom{n}{2}}$$

◀

Karger's Min-Cut Improved

```
procedure KargerSteinMinCut(G(V, E, c))
    n = |V|
    if n ≥ 6
        compute minimal cut deterministically
    else
        h = ⌈1 + n/√2⌉
        G/F_1 = Contract random edges in G until h nodes left
        (C_1, cost_1) = KargerSteinMinCut(G/F_1)
        G/F_2 = Contract random edges in G until h nodes left
        (C_2, cost_2) = KargerSteinMinCut(G/F_2)
        if cost_1 < cost_2 return (C_1, cost_1) else C_2, cost_2)
```

Theorem 9.15 (`KargerSteinMinCut` beats deterministic min-cut)

`KargerSteinMinCut` runs in time $O(n^2 \cdot \log(n))$ and finds a minimal cut with probability $\Omega(\frac{1}{\log(n)})$.