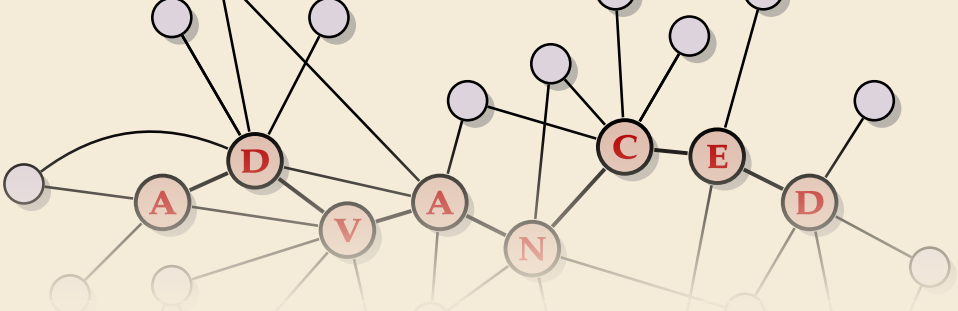

2 Complexity Theory Recap

22 April 2025

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CS627 (Summer 2025)
Philipps-Universität Marburg
version 2025-05-12 18:18

Outline

2 Complexity Theory Recap

2.1 P and NP Informally

Hard problems

- Some algorithmic problems are **"hard nuts" to crack**.
  - e. g., the *Traveling Salesperson Problem (TSP)*:

    Given: n cities $S_1, \ldots, S_n$,
    all $n(n-1)$ pairwise
    distances $d(S_i, S_j) \in \mathbb{N} \quad (i \neq j)$

    

    Goal: Shortest round trip through all cities

always exact, always correct — polytime

- no general, efficient algorithm known!
  (despite decades of intensive research . . .)

permutation

$\{1, \ldots, n\}$

$S_{i_1}, S_{i_2}, \ldots, S_{i_n} \qquad \{i_1, \ldots, i_n\} = \{n\}$

$\text{cost}(\,) = \sum_{j=1}^{n-1} d(S_{i_j}, S_{i_{j+1}}) + d(S_{i_n}, S_{i_1})$

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  ⇝ It *seems* as if there is no efficient algorithm for TSP!

  - **But:** can we *prove* that?

  - Despite similarly intensive research: **No!** (not yet)

Doesn't sound like a shining example for theoretical computer science? . . . stay tuned!

United in incapacity

"I can't find an efficient algorithm, but neither can all these famous people."

Garey, Johnson 1979

Complexity Theory

- *Complexity theory* allows us to *compare* the *hardness* of algorithmic problems.

A: old problem
Consensus: hard

B: new problem
Status: unknown
(seems hard to *us* . . .)

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2. B is at least as hard as A,

then B is a hard nut, too!

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"reduce A to B"

$\leq_p$

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Formally:

efficient = polytime

1. A is NP-hard: probably $\nexists$ eff. alg. for A
2. $A \leq_p B$: $\exists$ eff. alg. for $B \implies \exists$ eff. alg. for A

$\rightsquigarrow$ B is NP-hart: probably $\nexists$ eff. alg. for B!

P and NP – Intuitive Synopsis

$$P = NP?$$

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- P = class of problems for which there is an algorithm A and a polynomial p such that A **solves** every instance I in time $O(p(|I|))$.
  - P for "polynomial" – i. e., all problems where a solution can be *found* by a (deterministic) algorithm in polynomial time.

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  - NP for "nondeterministically polynomial" – i. e., all problems where a solution can be *found* by a *nondeterministic* algorithm in polynomial time.
  - This is equivalent to the above characterization via verification.
- We know $\mathsf{P} \subseteq \mathsf{NP}$. We *think* $\mathsf{P} \subsetneq \mathsf{NP}$, i. e., $\mathsf{P} \neq \mathsf{NP}$.
  The question "P = NP?" is one of the famous **millenium problems** and arguably the **most important open problem of theoretical computer science**.

2.2 Models of Computation

Clicker Question

What is the cost of *adding* two (decimal) d-digit integers?
(For example, for $d = 5$, what is $45\,235 + 91\,342$?)

- **A** constant time
- **B** logarithmic in d
- **C** proportional to d
- **D** quadratic in d
- **E** no idea what you are talking about

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Mathematical Models of Computation

- complexity classes talk about sets of problems based upon whether they allow an algorithm of a certain cost
- in general, this depends on the allowable algorithms and their costs!

⇝ need to fix a machine model

Mathematical Models of Computation

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- in general, this depends on the allowable algorithms and their costs!

$\rightsquigarrow$ need to fix a machine model

A *machine model* decides

- what algorithms are possible
- how they are described (= programming language)
- what an execution *costs*

Goal: Machine models should be
detailed and powerful enough to reflect actual machines,
abstract enough to unify architectures,
simple enough to analyze.

Random Access Machines

Standard model for detailed complexity analysis:

Random access machine (RAM)

more detail in §2.2 of *Sequential and Parallel Algorithms and Data Structures* by Sanders, Mehlhorn, Dietzfelbinger, Dementiev

- unlimited *memory* $\texttt{MEM}[0], \texttt{MEM}[1], \texttt{MEM}[2], \ldots$
- fixed number of *registers* $R_1, \ldots, R_r$ (say $r = 100$)

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- memory cells $\texttt{MEM}[i]$ and registers R_i store w-bit integers, i. e., numbers in $[0..2^w - 1]$
  w is the word width/size; typically $\boxed{w \propto \lg n}$ $\rightsquigarrow$ $2^w \approx n$

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  w is the word width/size; typically $\boxed{w \propto \lg n} \rightsquigarrow 2^w \approx n$
- Instructions:
  - load & store: $R_i := \texttt{MEM}[R_j]$ $\texttt{MEM}[R_j] := R_i$
  - operations on registers: $R_k := R_i + R_j$ (arithmetic is *modulo* 2^w!)
    also $R_i - R_j$, $R_i \cdot R_j$, $R_i \operatorname{div} R_j$, $R_i \bmod R_j$
    C-style operations (bitwise and/or/xor, left/right shift)
  - conditional and unconditional jumps
- time cost: number of executed instructions
- space cost: total number of touched memory cells

RAM-Program Example

Example RAM program

```
1  // Assume: R_1 stores number N
2  // Assume: MEM[0..N) contains list of N numbers
3  R_2 := R_1;
4  R_3 := R_1 − 2;
5  R_4 := MEM[R_3];
6  R_5 := R_3 + 1;
7  R_6 := MEM[R_5];
8  if (R_4 ≤ R_6) goto line 11;
9  MEM[R_3] := R_6;
10 MEM[R_5] := R_4;
11 R_3 := R_3 − 1;
12 if (R_3 ≥ 0) goto line 5;
13 R_2 := R_2 − 1;
14 if (R_2 > 0) goto line 4;
15 // Done:
```

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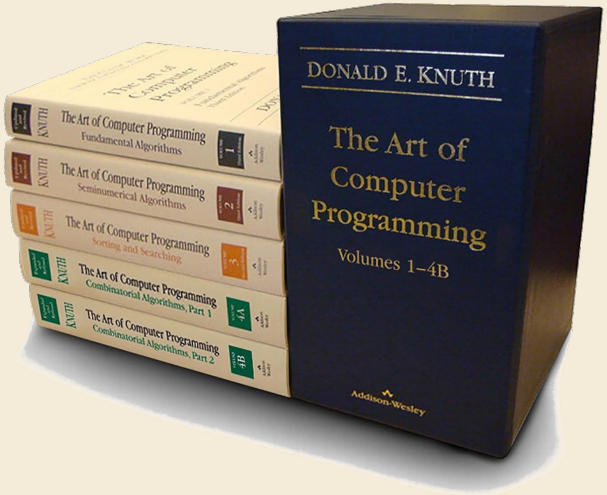

RAM-Program Example

Example RAM program

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12 **if** $(R_3 \geq 0)$ goto line 5;
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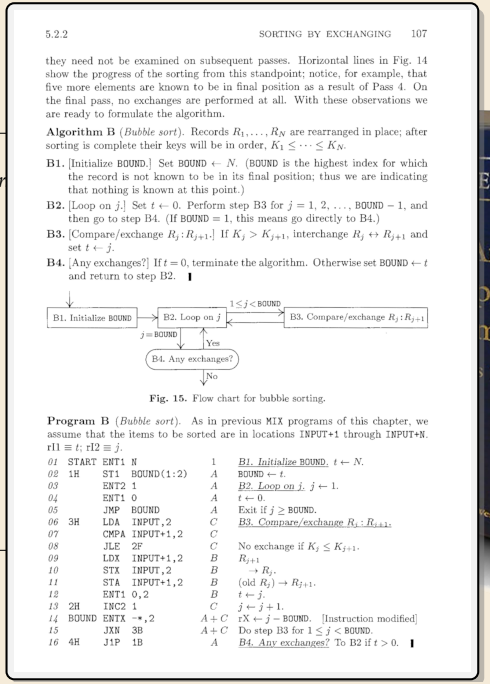
14 **if** $(R_2 > 0)$ goto line 4;
15 // *Done:* $\texttt{MEM}[0..N]$ *sorted*



5.2.2 SORTING BY EXCHANGING 107

they need not be examined on subsequent passes. Horizontal lines in Fig. 14 show the progress of the sorting from this standpoint; notice, for example, that five more elements are known to be in final position as a result of Pass 4. On the final pass, no exchanges are performed at all. With these observations we are ready to formulate the algorithm.

Algorithm B (*Bubble sort*). Records $R_1, \ldots, R_N$ are rearranged in place; after sorting is complete their keys will be in order, $K_1 \leq \cdots \leq K_N$.

B1. [Initialize BOUND.] Set BOUND $\leftarrow N$. (BOUND is the highest index for which the record is not known to be in its final position; thus we are indicating that nothing is known at this point.)

B2. [Loop on j.] Set $t \leftarrow 0$. Perform step B3 for $j = 1, 2, \ldots,$ BOUND $- 1$, and then go to step B4. (If BOUND $= 1$, this means go directly to B4.)

B3. [Compare/exchange $R_j : R_{j+1}$.] If $K_j > K_{j+1}$, interchange $R_j \leftrightarrow R_{j+1}$ and set $t \leftarrow j$.

B4. [Any exchanges?] If $t = 0$, terminate the algorithm. Otherwise set BOUND $\leftarrow t$ and return to step B2. ∎

Fig. 15. Flow chart for bubble sorting.

Program B (*Bubble sort*). As in previous MIX programs of this chapter, we assume that the items to be sorted are in locations INPUT+1 through INPUT+N. rI1 $\equiv t$; rI2 $\equiv j$.

```
01  START ENT1 N              1     B1. Initialize BOUND. t ← N.
02  1H    ST1  BOUND(1:2)     A     BOUND ← t.
03        ENT2 1              A     B2. Loop on j. j ← 1.
04        ENT1 0              A     t ← 0.
05        JMP  BOUND          A     Exit if j ≥ BOUND.
06  3H    LDA  INPUT,2        C     B3. Compare/exchange R_j : R_{j+1}:
07        CMPA INPUT+1,2      C
08        JLE  2F             C     No exchange if K_j ≤ K_{j+1}.
09        LDX  INPUT+1,2      B     R_{j+1}
10        STX  INPUT,2        B       → R_j.
11        STA  INPUT+1,2      B     (old R_j) → R_{j+1}.
12        ENT1 0,2            B     t ← j.
13  2H    INC2 1              C     j ← j + 1.
14  BOUND ENTX -*,2           A+C   rX ← j − BOUND.  [Instruction modified]
15        JXN  3B             A+C   Do step B3 for 1 ≤ j < BOUND.
16  4H    J1P  1B             A     B4. Any exchanges? To B2 if t > 0. ∎
```

2.3 Turing Machines

Keep it Simple, Stupid

- word-RAM (rather) realistic, but complicated
  - note that the machine has to grow with the inputs(!)
- for a coarse distinction of running time complexity, simpler models suffice
  - useful to reason about “all algorithms”
  - machine is fixed for all inputs sizes apart from storage for input

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Many models of computation . . .

word RAM

- μ-recursive function
- Turing machines (TM)
- counter machines
- λ-calculus
- While-programs
- any Turing-complete language
- quantum computers
- . . .

. . . with strong equivalences:

1. all proven to lead to the *same* set of computable functions
2. **Church-Turing thesis**:
   any formalization of "effectively computable" is equivalent in this sense
3. ***Extended Church-Turing Thesis:***
   . . . and can be simulated with *polynomial overhead* on a TM
   - true for all on left . . .
   - except theoretical quantum computers!
   - ignore them for now ← wake me when they exist

Turing Machines

- invented by ***Alan Turing*** in 1936 as formalization for "computable by hand"

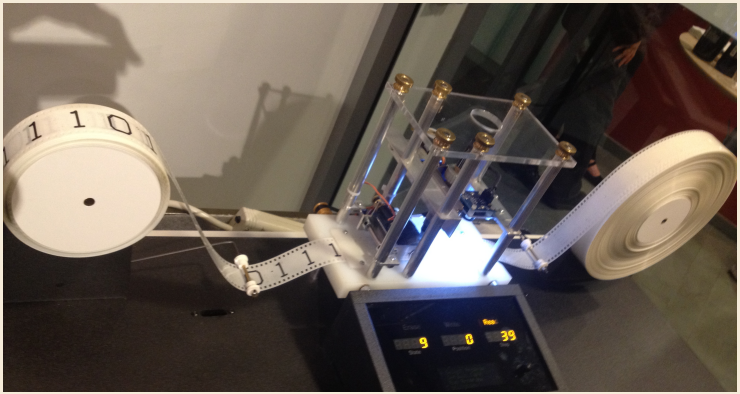
  In same paper, Turing proved undecidability of halting problem!

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

- *minimalistic* model of universal computer, but can be built:

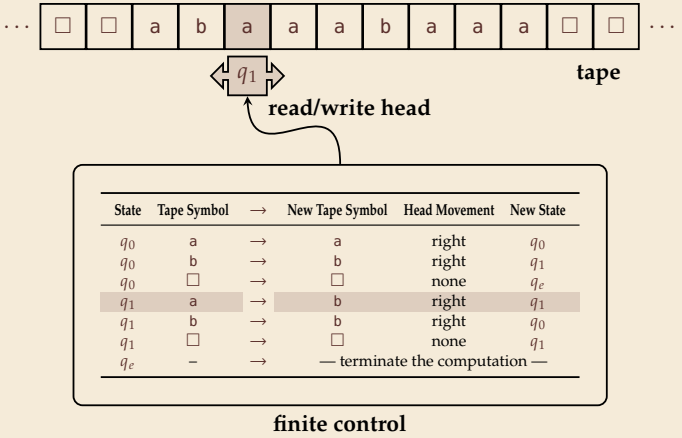


Turingmaschinen in Circulation

Turing Machines – Informal Recap

A *Turing machine* has

- a **finite control** via states
- an **input/output-tape**
  - unbounded length
  - initially contains input
  - all other cells contain "□"
- a **read/write head**
  - reads the current symbol
  - overwrites it with a new symbol
  - initially placed on beginning of input



Turing Machines – Formal Syntax

Definition 2.1 (Turing Machine (TM))

A *Turing machine* is a 7-tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, \square, q_{\text{halt}})$ with

- a finite set of *states* Q,
- an *input alphabet* Σ,
- a *tape alphabet* $\Gamma \supset \Sigma$,
- for deterministic TMs a *transition function* $\delta : (Q \setminus \{q_{\text{halt}}\}) \times \Gamma \to Q \times \Gamma \times \{L, R, N\}$
  for **non**deterministic TMs a *transition relation* $\delta : (Q \setminus \{q_{\text{halt}}\}) \times \Gamma \to 2^{Q \times \Gamma \times \{L,R,N\}}$
- an *initial state* $q_0 \in Q$,
- a *blank symbol* $\square \in \Gamma \setminus \Sigma$, and
- a halting state $q_{\text{halt}} \in Q$

$2^S = \{ S' : S' \subseteq S \}$

$\mathcal{P}(S)$ $\mathbb{P}(S)$

Turing Machine – Computation Step

- Each step of a computation of TM M has the form

  $\delta(q, a) = (q', b, d)$ resp. $\delta(q, a) \ni (q', b, d)$,

  with the semantics that
  - M is in state $q \neq q_{\text{halt}}$
  - the cell below the read/write head currently contains symbol a
  - M now changes (based on its finite control)
    - into state q',
    - writes b into the cell under the read/write head
    - and finally moves the read/write head in direction $d \in \{L, R, N\}$.
      (L = left, R = right, N = none (stay))

- for deterministic TM M, q and a uniquely determine this action;
  for nondeterministic TM, we may have several possible actions.

- to formally define an entire computation, we have to encode the tape contents as well

Turing Machines – Configurations

Definition 2.2 (TM Configuration)

A *configuration (config)* of a TM M is a string $C \in \Gamma^\star Q \Gamma^\star$. ◀

The semantics of a config $C = \alpha q \gamma$, $q \in Q$, is tape content $\alpha\beta$ and head at first symbol of β.

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Definition 2.3 (TM Computation Relation)

The *computation relation* $\vdash$ is defined on the set of configurations of a TM M as follows.

$$a_1 \dots a_m \, q \, b_1 \dots b_n \;\vdash\; \begin{cases} a_1 \dots a_m \quad q' \, c \, b_2 \dots b_n, & \delta(q, b_1) = (q', c, N),\ m \geq 0,\ n \geq 1, \\ a_1 \dots a_m \, c \;\; q' \, b_2 \dots b_n, & \delta(q, b_1) = (q', c, R),\ m \geq 0,\ n \geq 2, \\ a_1 \dots a_{m-1} \, q' \, a_m \, c \, b_2 \dots b_n, & \delta(q, b_1) = (q', c, L),\ m \geq 1,\ n \geq 1. \end{cases}$$

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For the boundary case $n = 1$ and direction right, we set

$$a_1 \dots a_m \, q \, b_1 \;\vdash\; a_1 \dots a_m \, c \, q' \, \square \quad \text{if } \delta(q, b_1) = (q', c, R),$$

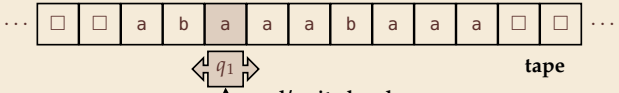
For $m = 0$ and direction left, we similarly have set

$$q \, b_1 \dots b_n \;\vdash\; q' \, \square c \, b_2 \dots b_n \quad \text{if } \delta(q, b_1) = (q', c, L).$$

Turing Machines – Configuration Example

Turing Machines – Configuration Example

Example:
For the shown TM M,
the current configuration is:
$C = \texttt{ab}\, q_1\, \texttt{aaabaaa}$



State	Tape Symbol	→	New Tape Symbol	Head Movement	New State
q_0	a	→	a	right	q_0
q_0	b	→	b	right	q_1
q_0	□	→	□	none	q_e
q_1	a	→	b	right	q_1
q_1	b	→	b	right	q_0
q_1	□	→	□	none	q_1
q_e	–	→	— terminate the computation —		

- TM Config $C = \alpha q \beta$ completely describes current state of computation
  - $\alpha\beta$ is the (non-blank) tape content
  - q is the current state of the TM
  - the read/write head is on the first symbol of β

Turing Machines – Computed Function

- With this setup, we can now formally define what a Turing machine computes.

Definition 2.4 (Function computed by a TM)

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \square, q_{\text{halt}})$ by a TM. The function computed by M on input $x \in \Sigma^\star$, written $M(x)$, is defined as
$M(x) = \{y : q_0 x \vdash^\star q_{\text{halt}} y\}$.
For deterministic TMs, we will also have $|M(x)| \leq 1$ and we write $M(x) = y$ for $M(x) = \{y\}$. ◀

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Definition 2.5 (Time and Space cost)

For a TM M and input $x \in \Sigma^*$, we define
$time_M(x) = \inf\{t : q_0 x \vdash^t q_{\text{halt}} y\} \cup \{0\}$.
We define $space_M(x) = \inf\{|\alpha\beta| : q_0 x \vdash^\star \alpha q \beta \vdash^\star q_{\text{halt}} y, |\alpha\beta|_\square \leq 2\} \cup \{0\}$. ◀

not important

$\square$ in $\alpha\beta$

- Note: *time* and *space* can be ∞ or 0 for nondeterministic TMs.

Turing Machines – Accepted Language

- Often convenient to use language acceptance instead of function computation.
- for deterministic TM, compute *characteristic function* of L: $\mathbb{1}_L(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$
- care needed for nondeterministic TM

Turing Machines – Accepted Language

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- care needed for nondeterministic TM

Definition 2.6 (Language of TM)

The language $\mathcal{L}(M)$ accepted by a TM M is defined as

$$\mathcal{L}(M) \quad = \quad \{w \in \Sigma^\star : 1 \in M(w)\}.$$

◀

$\rightsquigarrow$ nondeterministic TM accepts w iff some computation accepts w

Turing Machines – Several tapes

Remark 2.7 (k-tape TMs)

We only consider one-tap TMs here. In general, k-tape TMs can be faster.
However, any language accepted by a k-tape TM in time $f(n)$
is also be accepted by a 1-tape TM with running time $O(f^2(n))$.
The models are thus *polynomially equivalent*.

Turing Machines – Totality

- In complexity theory, we will restrict ourselves to TMs that always halt.

Definition 2.8 (Terminating TM)

A TM M is *always terminating / total* if there is a function $T : \mathbb{N} \to \mathbb{N}$ such that $q_0 x = C_0 \vdash C_1 \vdash \cdots \vdash C_t$ implies $t \leq T(|x|)$, and there is a y such that $q_0 x \vdash^\star q_{\text{halt}} y$. ◀

- Note that in a terminating TM, we always have $1 \leq time_M(x) \leq T(|x|)$.

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Lemma 2.9 (Time-constrained TM)

Given a (potentially nonterminiating) TM M and a function $T : \mathbb{N} \to \mathbb{N}$ **computable** in time $T(n)$, we can construct an *always terminating* TM M' that simulates M for $T(|x|)$ on x and outputs `TIMEOUT` if M has not terminated yet. Moreover, $time_{M'}(x) = O(T^2(|x|))$ for all $x \in \Sigma^\star$. ◀

Idea: M' computes $T(|x|)$ and writes $T(|x|)$ many (L) $\in \Gamma$

then simulate M step by step, then delete (L)

when no (L) left, output TIMEOUT

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$\rightsquigarrow$ If we are only interested in the (non)existence of a polynomial-time TM, we can restriction ourselves to total TMs that will never `TIMEOUT`.

Models of Computation – Summary

- Concrete model such as TMs useful for some proofs
- Often, details do not matter as long as models are polynomially equivalent
  - Note: TM always means we are in the logarithmic cost model for arithmetic operations
- In the following, discuss more abstract notion of "algorithm"
  (fine to substitute by TM in each case)

2.4 The Classes P und NP

Worst Case Complexity

Definition 2.10 (Time and Space Complexity – Generic)

Let Σ_I and Σ_O two alphabets and A an algorithm implementing a **total** mapping $\Sigma_I^\star \to \Sigma_O^\star$. Then for each $x \in \Sigma_I^\star$ we denote by $time_A(x)$ (resp. $space_A(x)$) the logarithmic time complexity (resp. logarithmic space complexity) for A on x. ◀

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Definition 2.11 (Worst-Case Complexity)

Let Σ_I and Σ_O be two alphabets and A an algorithm implementing a **total** mapping $\Sigma_I^\star \to \Sigma_O^\star$. The *worst case time complexity of* A is the function $Time_A : \mathbb{N} \to \mathbb{N}$ with

$$Time_A(n) \quad = \quad \max\{time_A(x) : x \in \Sigma_I^{\leq n}\},$$

for each $n \in \mathbb{N}$. The *worst case space complexity of* A is given by function $Space_A : \mathbb{N} \to \mathbb{N}$ with

$$Space_A(n) \quad = \quad \max\{space_A(x) : x \in \Sigma_I^n\}.$$ ◀

Decision Problems = Languages

Definition 2.12 (Decision Problem and Algorithms)

A *decision problem* is given by $P = (L, U, \Sigma)$ for Σ an alphabet and $L \subseteq U \subseteq \Sigma^\star$. An algorithm A *solves (decides)* decision problem P, if for all $x \in U$

1. $A(x) = 1$ for $x \in L$, and
2. $A(x) = 0$ for $x \in U \setminus L$ (i. e., $x \notin L$)

holds. Here $A(x)$ denotes the output of A on input x.
If $U = \Sigma^\star$ holds we denote P briefly by (L, Σ). ◀

$\rightsquigarrow$ A computes a total function $A : U \to \{0, 1\}$,
the *characteristic function* $\mathbb{1}_L : U \to \{0, 1\}$ of language L.
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We restrict our attention to ***decision problems***:
Given: $w \in \Sigma^\star$.
Goal: Is $w \in L$?

Example:
w is an encoding of an instance of the traveling salesperson problem **and** a threshold D

$L = \{w : w \text{ encodes instance w/ opt. round trip length} \leq D\}$

Optimal Algorithms

Definition 2.13 (Upper/Lower Bounds, Optimal Algorithms)

Let U be an algorithmic problem and f, g functions $\mathbb{N}_0 \to \mathbb{R}^+$.

- We call $O(g(n))$ an *upper bound for time complexity of U*
  if there is an algorithm A that solves U in time $Time_A(n) \in O(g(n))$.
- We say $\Omega(f(n))$ is a *lower bound for time complexity of U*
  if every algorithm A that solves U needs time $Time_A(n) \in \Omega(f(n))$.
- An algorithm A is called *optimal for U*
  if $Time_A(n) \in O(g(n))$ and $\Omega(g(n))$ is a lower bound for the time complexity of U.

Running Time

Definition 2.14 (time classes)

For function $f : \mathbb{N} \to \mathbb{N}$, the class $TIME(f(n))$ is the set of all languages A, for which there is a *deterministic* Turing machine M with $\mathcal{L}(M) = A$ and $time_M(w) \leq f(|w|)$ für alle $w \in \Sigma^\star$. ◀

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Definition 2.15 (P, tractable)

We define the class of languages P decidable in polynomial time by

$$\mathsf{P} := \bigcup_{p \text{ polynomial}} TIME(p(n)). = \bigcup_{c \in \mathbb{N}} TIME(n^c)$$

A language (a decision problem) $L \in \mathsf{P}$ is called *tractable / efficiently decidable*. ◀

Nondeterministic Running Time

Recall:

- A nondeterministic Turing machine / algorithm M accepts L ($\mathcal{L}(M) = L$) if for all $x \in L$ there is at least one computation of M which accepts x and for all $y \notin L$ every computation of M rejects y.
- We only consider always-terminating Turing machines.
- The running time $time_M(x)$ of M on x is given by the longest computation of M on x.

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Definition 2.16 (NTIME, NP)

For function $f : \mathbb{N} \to \mathbb{N}$, the class $NTIME(f(n))$ is the set of all languages A, for which there is a ***non****deterministic* Turing machine M with $\mathcal{L}(M) = A$ and $time_M(w) \leq f(|w|)$ für alle $w \in \Sigma^\star$.

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The class of languages NP is defined by

$$\mathsf{NP} := \bigcup_{p \text{ polynomial}} NTIME(p(n)).$$

◀

2.5 Nondeterminism = Verification

Nondeterminism?

- The original definition of NP via nondeterministic Turing machines is not very intuitive.
- There is an equivalent characterization that is usually more convenient to use: Certificates and verifiers.

Polynomially verifiable

Definition 2.17 (Certificates, Verifier, VP)

Let $L \subseteq \Sigma^\star$ be a language.

- An algorithm A acting on inputs from $\Sigma^\star \times \{0,1\}^\star$ is called *verifier for* L (notation $L = \mathcal{V}(A)$), if
$$L = \{w \in \Sigma^\star : \exists c \in \{0,1\}^\star \; A(w,c) = 1\}.$$
If A accepts input (w,c) we say c is *proof* or *certificate* for $w \in L$.
- A verifier A for L is a *polynomial-time verifier* if there is a $d \in \mathbb{N}$ such that for all $w \in L$, there is a proof c (for $w \in L$) with $time_A(w,c) \in O(|w|^d)$.
- We define the class of polynomially verifiable languages VP by
$$\mathsf{VP} = \{\mathcal{V}(A) : A \text{ is polynomial time verifier}\}.$$

Ex $L = \{$ TSP Yes instances , d , D $\}$

Nondeterminism ↔ certificate

Theorem 2.18
NP = VP.

Proof: • NP ⊆ VP

$\exists c$ $Time_M(n) = O(n^c)$

$L \in NP \Rightarrow \exists$ nondet. TM M : $\mathcal{L}(M) = L$, polytime

Construct verifier A, $L \subseteq \Sigma^*$

$(x, c) \in \Sigma^* \times \{0,1\}^*$

A uses c as guide to pick nondeterministic computation

at each point in time, M has $|\delta(q, a)| \leq C$ options

use $\lceil \lg C \rceil$ bits in c to select one step

M accepts x $\Leftrightarrow$ $\exists$ accepting computation $(q_0 x \vdash^* q_{halt} 1)$ for x

$\Leftarrow$ $\exists c$ that guides A to accept (x, c)

$\Rightarrow V(A) = L \Rightarrow L \in VP$

- $VP \subseteq NP$

  start with polytime verifier A for $L \in VP$

  construct nondet. TM M

  1. M "guesses" nondeterministically binary string c
  2. simulate A

  $\mathcal{L}(M) = L \Rightarrow L \in NP$ □

$TSP \in NP = VP$ check length of given tour $\in D$

easy ☺

2.6 Karp-Reductions und NP-Completeness

Recap

- We restrict our attention to ***decision problems***:
  Given: $w \in \Sigma^\star$.
  Goal: Is $w \in L$?

Example:
w is an encoding of an instance of the traveling salesperson problem **and** a threshold D

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$\rightsquigarrow$ problems = (formal) languages $L \subseteq \Sigma^\star$

- problem instance = word $w \in \Sigma^\star$
- $w \in \Sigma^\star$ is a *Yes instance* if $w \in L$, otherwise a *No instance*

Recap

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⇝ problems = (formal) languages $L \subseteq \Sigma^\star$

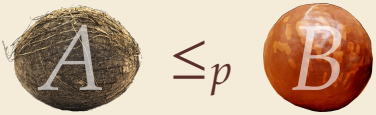
- problem instance = word $w \in \Sigma^\star$
- $w \in \Sigma^\star$ is a *Yes instance* if $w \in L$, otherwise a *No instance*

⇝ For problems on structures, e. g., graphs, we need an *encoding* of the instance as a string.
(often simple; standard data structures do the trick)

⇝ input size n of instance = length of the *encoding* of instance

⇝ all running times are worst case over instances of encoding length n

Karp Reductions



A: old problem
Consensus: hard

B: new problem
Status: unknown
(seems hard to *us* . . .)

- **Goal:** Show that B is at least as hard as A.

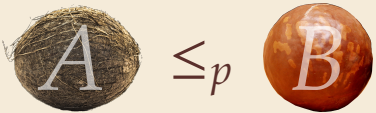
short for: deterministic TM M w/ $Time_M(n) = O(n^k)$ for constant k.

Assume there *was* a polytime algorithm M für B.
Solve A using M (in polytime).

$\rightsquigarrow$ polytime algo for B implies polytime algo for A

$\rightsquigarrow$ B at least as hard as A

Karp Reductions



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$\rightsquigarrow$ polytime algo for B implies polytime algo for A

$\rightsquigarrow$ B at least as hard as A

Formally: (strong notion than intuition above!)

Definition 2.19 (polytime reduction, $\leq_p$)

Let $A \subseteq \Sigma^\star$ and $B \subseteq \Gamma^\star$ be languages (decision problems).
A is *polytime reducible to* B – written $A \leq_p B$ – if there is a total function $g : \Sigma^\star \to \Gamma^\star$, computable in polynomial time, with

$$\forall w \in \Sigma^\star \; : \; w \in A \Leftrightarrow g(w) \in B.$$

◀

- This type of reduction is called a *Karp-reduction*
- It is more restrictive than our intuitive version would need, but allows finer complexity classification (NP and co-NP)

Implication of Reductions

Lemma 2.20 (Membership reduction)

If $A \leq_p B$ and $B \in \mathsf{P}$ (resp. $B \in \mathsf{NP}$), then $A \in \mathsf{P}$ (resp. $A \in \mathsf{NP}$). ◀

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Proof:

Since $B \in \mathsf{P}$, there is polytime TM M with $\mathcal{L}(M) = B$.

Since $A \leq_p B$, there further is polytime TM g mit $\forall w : w \in A \Leftrightarrow g(w) \in B \quad (*)$.

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first simulate g on input w, then simulate M on $g(w)$.

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Since $A \leq_p B$, there further is polytime TM g mit $\forall w : w \in A \Leftrightarrow g(w) \in B \quad (*)$.
We construct TM M' for A:
first simulate g on input w, then simulate M on $g(w)$.

Since M and g are polytime TMs, so is M', and $\mathcal{L}(M') = A$ (since $(*)$).
$\rightsquigarrow \quad A \in \mathsf{P}$.

(The version with $B \in \mathsf{NP}$ is similar, just using nondeterministic polytime). ■

NP Completeness

Definition 2.21 (NP-hard, NP-complete)

A language A is called NP-*hard*, if we have for **all** languages $L \in \mathsf{NP}$ that $L \leq_p A$.
A language A is called NP-*complete*, if A is NP-hard and $A \in \mathsf{NP}$. ◀

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Proof:

$\Rightarrow$ Let $L \in \mathsf{NP}$ be arbitrary.
Since A is (by assumption) NP-hard, we have $L \leq_p A$.
Since $A \in \mathsf{P}$, by membership reduction (Lemma 2.20) also $L \in \mathsf{P}$.
Since L was an *arbitrary* language from NP, $\mathsf{P} = \mathsf{NP}$ follows.

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Since L was an *arbitrary* language from NP, $\mathsf{P} = \mathsf{NP}$ follows.

⇐ Let conversely $\mathsf{P} = \mathsf{NP}$.
Then, by assumption, $A \in \mathsf{NP} = \mathsf{P}$. ■

Implications of NP-Completeness

- NP-completeness tells us a lot about a problem!

But shall we possible prove $L \leq_p A$ *for all* possible $L \in \mathsf{NP}$?

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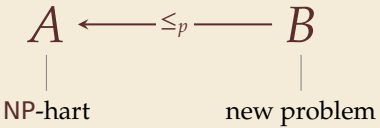
  (proof is similar to membership-reduction lemma)



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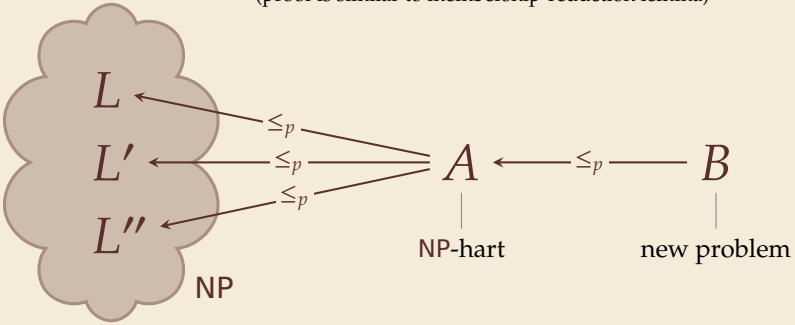

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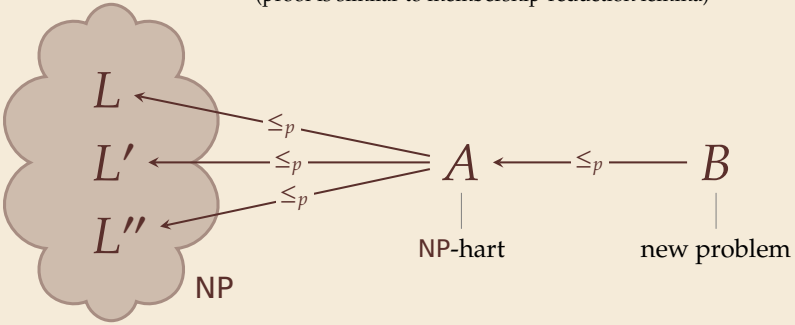

A NP-hard and $A \leq_p B$
$\rightsquigarrow$ B NP-hard

Clicker Question

What is the value of the formula φ under the variable assignment V?

$$\varphi = x_1 \wedge (\neg x_2 \vee x_3)$$
$$V = \{x_1 \mapsto \mathit{true},\ x_2 \mapsto \mathit{true},\ x_3 \mapsto \mathit{false}\}$$

A satisfiable

B *true*

C *false*

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→ sli.do/cs627

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→ *sli.do/cs627*

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Are there any *at all?*

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Theorem 2.23 (Cook-Levin)
SAT is NP-complete. ◀

- SAT is the *satisfiability problem* of propositional logic:
  Given: Boolean (propositional logic) formula φ over variables $x_1, \ldots, x_n$.
  Goal: Is there a variable assignment $V : \{x_1, \ldots, x_n\} \to \{true, false\}$, so that φ evaluates under V to *true*?

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Proof (Theorem 2.23):
Idea: Given any nondeterministic TM M for an arbitrary language $L \in$ NP, construct from a word w a formula $\varphi(w)$, which exactly encodes all valid computations of M.
Variables $x_{q,t}$: Is M be in state q at time t?
$y_{c,i,t}$: Does tape cell i contain char c at time t?
$z_{i,t}$ Does read/write head stand at position i at time t?
$\rightsquigarrow$ $\varphi(w)$ satisfiable iff M accepts w.

(for details, see, e. g., §2.3 of S. Arora and B. Barak: *Computational Complexity: A Modern Approach*) ■

2.7 Example of an NP-completeness proof

3SAT

- Let's do a more typical full example.

3SAT

- Let's do a more typical full example.
- Need one more NP-complete problem first

Definition 2.24 (3SAT)

Given: A Boolean formula φ in *3-CNF*:
conjunctive normal form with *at most* 3 literals per clause

Goal: Is there an assignment V of the variables in φ, so that φ evaluates to *true*?
(a.k.a. Is φ *satisfiable*?)

Example:

literal clause

$(x_1 \vee \neg x_3 \vee x_2) \wedge (\neg x_3 \vee x_4 \vee \neg x_5) \wedge (x_5 \vee \neg x_5 \vee \neg x_1) \wedge (x_1 \vee x_3 \vee x_5)$

satisfiable, e. g., via $x_1 \mapsto \mathit{true}$, $x_5 \mapsto \mathit{false}$ (other variables arbitrary)

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Theorem 2.25 (3SAT)

SAT $\leq_p$ 3SAT and 3SAT $\in$ NP. ◀

Corollary 2.26 (3SAT)

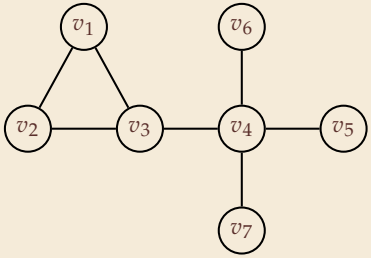
3SAT is NP-complete. ◀

Vertex Cover

Definition 2.27 (VertexCover)

Given: A (simple, undirected) graph $G = (V, E), E \subseteq \binom{V}{2}$, threshold k.

Goal: $\exists S \subseteq V$ with $|S| \leq k$, such that $\forall e \in E : S \cap e \neq \emptyset$? ◀



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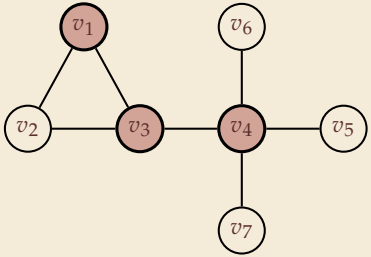

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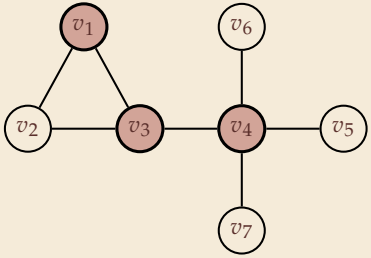

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Theorem 2.28 (VertexCover hard)

VertexCover is NP-complete. ◀

Proof:

We will prove (i) VertexCover $\in$ VP and (ii) 3SAT $\leq_p$ VertexCover.

$\rightsquigarrow$ Theorem 2.28 (since 3SAT is NP-complete and VP = NP).

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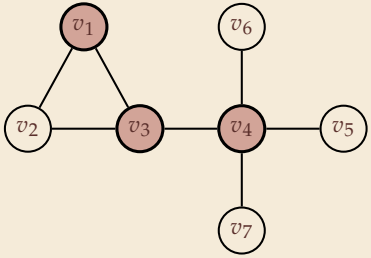

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(i) certificate = S; verifier whether all edges $e \in E$ are covered and whether $|S| \leq k$

$\rightsquigarrow$ clearly doable in polytime $\rightsquigarrow$ VertexCover $\in$ VP.

(ii) 3SAT $\leq_p$ VertexCover

Proof:

- Intuition: Express 3SAT instance as a VertexCover instance.

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- Intuition: Express 3SAT instance as a VertexCover instance.
- So, let φ be an arbitrary formula in 3-CNF over variables $x_1, \ldots, x_m$

$\rightsquigarrow$ φ has the form $\underbrace{(l_{1,1} \vee l_{1,2} \vee l_{1,3})}_{C_1} \wedge \underbrace{(l_{2,1} \vee l_{2,2} \vee l_{2,3})}_{C_2} \wedge \cdots \wedge \underbrace{(l_{n,1} \vee l_{n,2} \vee l_{n,3})}_{C_n}$,
with $l_{i,j} \in \{x_1, \neg x_1, \ldots, x_m, \neg x_m\}$ for $i = 1, \ldots, n$ and $j = 1, 2, 3$.

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- Define a graph $G = (V, E)$ via

$$V = \{L_{i,j} : i = 1, \ldots, n;\ j = 1, 2, 3\}$$
$$E = \left\{\{L_{i,j}, L_{p,q}\} : l_{i,j} \equiv \neg l_{p,q}\right\} \cup \left\{\{L_{i,1}, L_{i,2}\}, \{L_{i,2}, L_{i,3}\}, \{L_{i,3}, L_{i,1}\} : i = 1, \ldots, n\right\}$$

We "draw" a vertex for every literal of a clause. We connect them if
(a) they are literals in the same clause or (b) they are negations of each other

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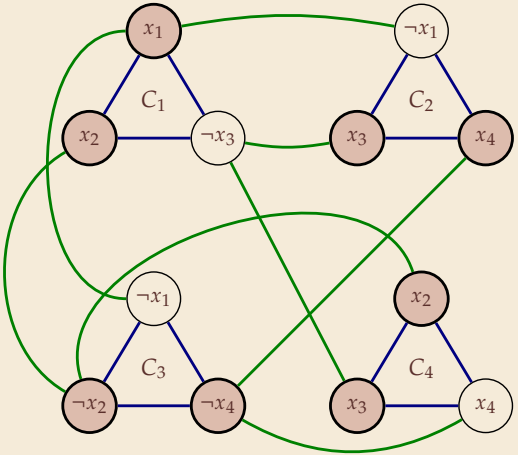
$\rightsquigarrow$ **Claim:** φ satisfiable $\iff$ G has vertex cover of size $\leq 2n$.

(ii) 3SAT $\leq_p$ VertexCover – Example

$\varphi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge (x_2 \vee x_3 \vee x_4)$



Set S (a VC of size $2n$)

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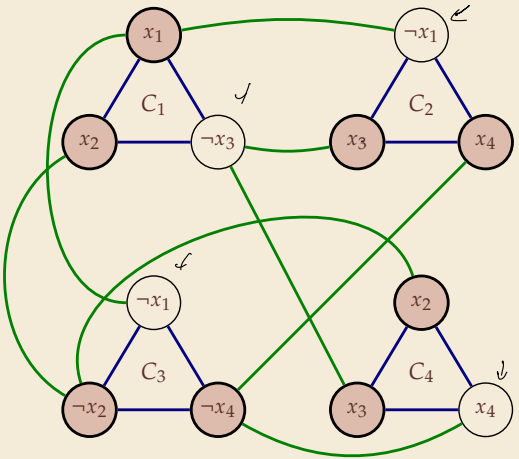

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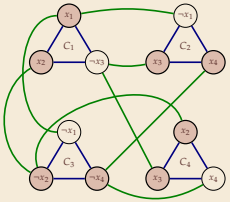
- **Idea:** Vertices ***not in*** vertex cover S define a variable assignment.
  - Cannot be contradictory, otherwise "negation"-edge not covered.
  - Must take ≥ 2 vertices per clause into S (otherwise triangle not covered)
  - $\rightsquigarrow$ $|S| \geq 2n$ for every vertex cover.
- In the example:
  - Fat vertices form a vertex cover for G
  - corresponding assignment: $V = \{x_1 \mapsto 0, x_2 \mapsto 0, x_3 \mapsto 0, x_4 \mapsto 1\}$ ($0 \mathrel{\widehat{=}} \textit{false}, 1 \mathrel{\widehat{=}} \textit{true}$)
  - $\rightsquigarrow$ φ satisfiable

(ii) 3SAT $\leq_p$ VertexCover – Correctness Proof

Claim: φ satisfiable $\iff$ G has VC S of size $|S| \leq 2n$.



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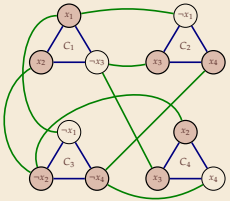

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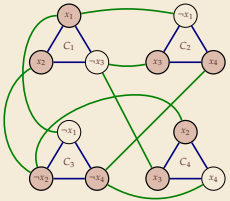

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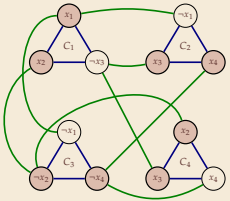

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Remaining edges have the form $\{x, \neg x\}$.
If such an edge remained uncovered by S, we would have that both x and $\neg x$ are satisfied literals $\lightning$
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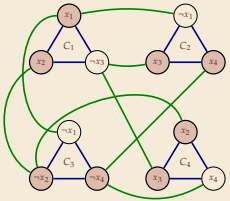

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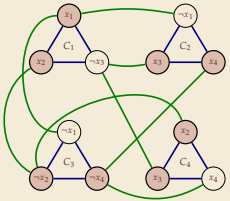

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$\Leftarrow$ Given a VC S of G with $|S| \leq 2n$.
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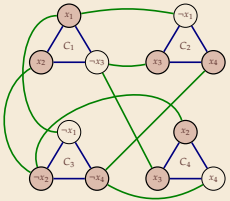

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Define assignment V so that all literals *not* in S are satisfied.
(Variables which are not assigned a value via this procedure can be assigned an arbitrary value.)
V is well defined, since $\{x, \neg x\}$-edges must be covered.

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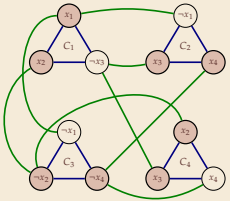

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Define assignment V so that all literals *not* in S are satisfied.
(Variables which are not assigned a value via this procedure can be assigned an arbitrary value.)
V is well defined, since $\{x, \neg x\}$-edges must be covered.
Moreover, V makes φ *true*:
from every clause, at least one literal is satisfied since $|S| = 2n$.
$\rightsquigarrow$ φ satisfiable. ∎

(ii) 3SAT $\leq_p$ VertexCover – Running Time

- Construction of G upon input φ can easily be done in polytime
  - $|V| = O(n)$, $|E| = O(n^2)$
  - Construction of E in time $O(n^2)$ easy to do, e. g., on RAM $\rightsquigarrow$ $\exists$ polytime TM.

$\rightsquigarrow$ 3SAT $\leq_p$ VertexCover. ■

2.8 Important NP-Complete Problems

Further NP-complete problems [1]

Apart from SAT, 3SAT, and VertexCover, here are some of the most useful NP-complete problems.

Definition 2.29 (Dominating Set)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$
Question: $\exists V' \subset V \;:\; |V'| \leq k \;\wedge\; \forall v \in V : \big(v \in V' \;\vee\; \exists u \in N(v) : u \in V'\big)$ ◀

Definition 2.30 (Hamiltonian Cycle)

Given: graph $G = (V, E)$ (directed and undirected version)
Question: Is there a vertex-simple cycle in G of length $|V|$? ◀

Definition 2.31 (Clique)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$
Question: $\exists V' \subset V \;:\; |V'| \geq k \;\wedge\; \forall u, v \in V' : \{u, v\} \in E$ ◀

Definition 2.32 (Independent Set)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$
Question: $\exists V' \subset V \;:\; |V'| \geq k \;\wedge\; \forall u, v \in V' : \{u, v\} \notin E$ ◀

Further NP-complete problems [2]

Definition 2.33 (Traveling Salesperson (TSP))

Given: distance matrix $D \in \mathbb{N}^{n \times n}$ and $k \in \mathbb{N}$
Question: Is there a permutation $\pi : [n] \to [n]$ with $\sum_{i=1}^{n-1} D_{\pi(i),\pi(i+1)} + D_{\pi(n),\pi(1)} \leq k$? ◀

Definition 2.34 (Graph Coloring)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$
Question: $\exists c : V \to [k] \ : \ \forall \{u, v\} \in E : c(u) \neq c(v)$? ◀

Definition 2.35 (Set Cover)

Given: $n \in \mathbb{N}$, sets $S_1, \ldots, S_m \subseteq [n]$ and $k \in \mathbb{N}$
Question: $\exists I \subseteq [m] \ : \ \bigcup_{i \in I} S_i = [n] \ \wedge \ |I| \leq k$? ◀

Definition 2.36 (Weighted Set Cover)

Given: $n \in \mathbb{N}$, sets $S_1, \ldots, S_m \subseteq [n]$, costs $c_1, \ldots, c_m \in \mathbb{N}_0$ and $k \in \mathbb{N}$
Question: $\exists I \subseteq [m] \ : \ \bigcup_{i \in I} S_i = [n] \ \wedge \ \sum_{i \in I} c_i \leq k$? ◀

Further hard problems [3]

Definition 2.37 (Closest String)

Given: $s_1, \ldots, s_n \in \Sigma^m$ and $k \in \mathbb{N}$

Question: $\exists s \in \Sigma^m \;:\; \forall i \in [n] : d_H(s, s_i) \leq k$? (d_H Hamming-distance) ◀

Definition 2.38 (Max Cut)

Given: graph $G = (V, E)$ and $k \in \mathbb{N}$

Question: $\exists C \subset V \;:\; \left|E \cap \{\{u, v\} \mid u \in C, v \notin C\}\right| \geq k$? ◀

Definition 2.39 (Exact Cover)

Given: $n \in \mathbb{N}$, sets $S_1, \ldots, S_m \subseteq [n]$

Question: $\exists I \subseteq [m] \;:\; \bigcup_{i \in I} S_i = [n] \;\wedge\; \sum_{i \in I} |S_i| = n$? ◀

Further hard problems [4]

Definition 2.40 (Subset Sum)

Given: $x_1, \ldots, x_n \in \mathbb{Z}$

Question: $\exists I \subseteq [n] \ : \ I \neq \emptyset \wedge \sum_{i \in I} x_i = 0$? ◀

Definition 2.41 ((0/1) Knapsack)

Given: $w_1, \ldots, w_n \in \mathbb{N}$, $v_1, \ldots, v_n \in \mathbb{N}$ and $b, k \in \mathbb{N}$

Question: $\exists I \subseteq [n] \ : \ \sum_{i \in I} w_i \leq b \ \wedge \ \sum_{i \in I} v_i \geq k$? ◀

Definition 2.42 (Bin Packing)

Given: $w_1, \ldots, w_n \in \mathbb{N}$, $b \in \mathbb{N}$, $k \in \mathbb{N}$

Question: $\exists a : [n] \to [k] \ : \ \forall j \in [k] : \sum_{\substack{i=1,\ldots,n \\ a[i]=j}} w_i \ \leq \ b$? ◀

Definition 2.43 (0/1 Integer Programming)

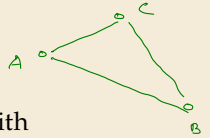
Given: integer linear program (ILP) $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$

Question: Is there $x \in \{0,1\}^n$ with $Ax \leq b$ and $c^T x \geq k$? ◀

2.9 Optimization Problems

Optimization Problems

metric TSP



$D(A,B) \leq D(A,C) + D(C,B)$

Definition 2.44 (Optimization Problem)

An *optimization problem* is given by 7-tuple $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ with

1. Σ_I an alphabet (called input alphabet),
2. Σ_O an alphabet (called output alphabet),
3. $L \subseteq \Sigma_I^\star$ the language of allowable problem instances (for which U is well-defined), — distance matrices
4. $L_I \subseteq L$ the language of actual problem instances for U (for those we want to determine U's complexity), — metric distance matrices
5. $M : L \to 2^{\Sigma_O^\star}$ and with $x \in L$, $M(x)$ is the set of all feasible solutions for x. — permutations of cities
6. *cost* is a cost function, which assigns for $x \in L$ each pair (u, x) with $u \in M(x)$ a positive real number,
7. $goal \in \{\min, \max\}$.

◀

Optimal Solutions

Definition 2.45 (Optimal Solutions, Solution Algorithms)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ an optimization problem. For each $x \in L_I$ a feasible solution $y \in M(x)$ is called *optimal for x and U*, if

$$cost(y, x) \quad = \quad goal\{cost(z, x) \mid z \in M(x)\}. \qquad \blacktriangleleft$$

An algorithm A is *consistent with U* if $A(x) \in M(x)$ for all $x \in L_I$.

We say *algorithm B solves U*, if

1. B is consistent with U and
2. for all $x \in L_I$, $B(x)$ is optimal for x and U. $\blacktriangleleft$

Optimization Problems – Examples

Natural examples: Problems above with an input parameter k.

Less immediate example:

$C_i = (x_i \vee \neg x_j \vee x_k \dots)$

Definition 2.46 (Max-SAT)

Given: CNF-Formula $\phi = C_1 \wedge \cdots \wedge C_m$ over variables $x_1, \dots, x_n$
Allowable (=Actual) Instances: encodings of ϕ
$M(\phi) = \{0,1\}^n$ (variable assignments)
$cost(u, x)$: # of satisfied clauses in u under given assignment x
$goal = \max$

Classes of Optimization Problems

Definition 2.47 (NPO)

NPO is the class if optimization problems $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ with

1. $L_I \in \mathsf{P}$,
2. there is a polynomial p_U with
   a) $\forall x \in L_I \; \forall y \in M(x) : |y| \leq p_U(|x|)$ and
   b) there is a polynomial time algorithm which for all $y \in \Sigma_O^\star$, $x \in L_I$ with $|y| \leq p_U(|x|)$ decides whether $y \in M(x)$ holds, and
3. function $cost$ can be computed in polynomial time. ◀

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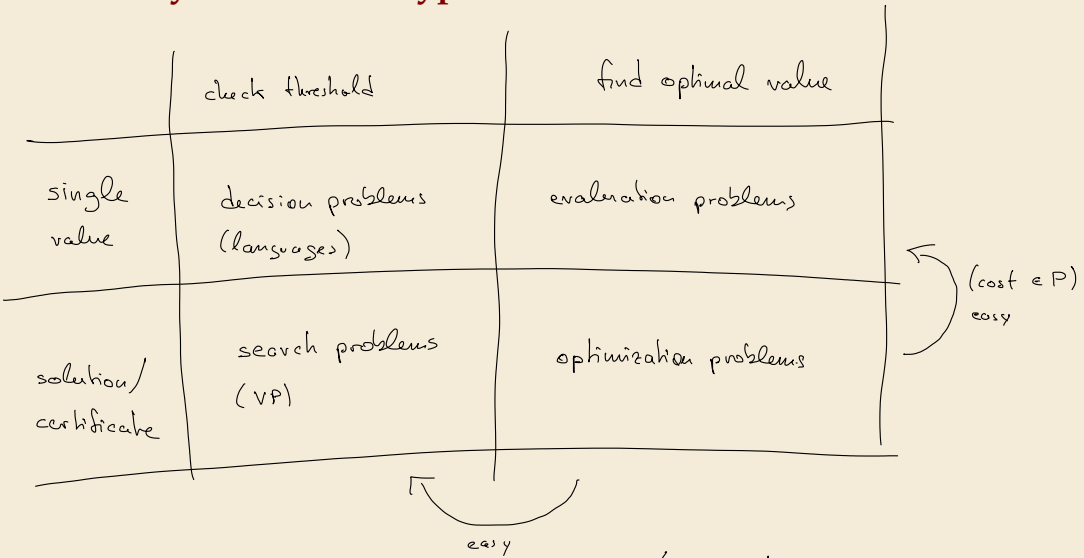
Definition 2.48 (PO)

PO is the class of optimization problems $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ with

1. $U \in \mathsf{NPO}$, and
2. there is an algorithm of polynomial time complexity which for all $x \in L_I$ computes an optimal solution for x and U. ◀

Glossary of Problem Types



- ↘ threshold to opt ? : binary search ✓ cost ∈ ℕ
- ↙ decision to search : unclear in general -- often possible

From Optimization to Decision

Definition 2.49 (Threshold Languages)

Let $U = (\Sigma_I, \Sigma_O, L, L_I, M, cost, goal)$ an optimization problem, $U \in \mathsf{NPO}$.
For $Opt_U(x)$ the cost of an optimal solutions for x and U we define the *threshold language for* U as

$$Lang_U = \begin{cases} \{(x,k) \in L_I \times \{0,1\}^\star \mid Opt_U(x) \leq k_2\}, & \text{if } goal = \min, \\ \{(x,k) \in L_I \times \{0,1\}^\star \mid Opt_U(x) \geq k_2\}, & \text{if } goal = \max. \end{cases}$$

k as binary number

We say U *is* NP*-hard*, if $Lang_U$ is NP-hard. ◀

Hardness of Optimization

Corollary 2.50 (Optimization is harder than Threshold)

Let U an optimization problem.
If $Lang_U$ is NP-hard and if $\mathsf{P} \neq \mathsf{NP}$ holds, we have $U \notin \mathsf{PO}$. ◀

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↯ $Lang_U$ NP-hard $\rightsquigarrow$ $U \notin$ PO. ■

Max-SAT is hard

Corollary 2.51 (Max-SAT is hard)

MAX-SAT is NP-hard. ◀

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We show: $3\text{SAT} \leq_p Lang_{\text{MAX-SAT}}$.

Given a 3SAT instance x encoding a formula with m clauses, output (x, m).

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Proof:

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Given a 3SAT instance x encoding a formula with m clauses, output (x, m).

$x \in 3\text{SAT} \iff x$ is satisfiable $\iff m$ clauses of x satisfiable $\iff (x, m) \in Lang_{\text{MAX-SAT}}$. ■

Summary

- We have formalized the classic notion of intractable problems.
  - What is running time, what is "polytime"?
  - Decision problems $\leftrightarrow$ (formal) languages
  - P, NP via Turing machines $\leftrightarrow$ certificates and verifiers
  - For the typical case of optimization problems,
    there are different versions of the problem,
    but (in)tractability typically carries over.

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$\rightsquigarrow$ We can mathematically prove a problem is intractable (NP-hard).

. . . but how can we tackle hard problems anyway?